Math 1210-23

Notes of 3/20/24

Sum Notation

- Let's review and practice the Sum Notation:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

- Some examples:

$$\int_0^1 x^4 dx = \int_0^1 t^4 dt$$

$$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2 = 14$$

$$\sum_{i=1}^{100} i = 5050$$

$$\sum_{i=1}^{100} \sin i - \sum_{j=1}^{100} \sin j = 0$$

- The lower limit of the sum may not be 1. How would you define

$$\sum_{i=2}^{5} i = 2+3+4+5 = 14$$

$$\sum_{i=-100}^{100} i^5 = 0$$

Properties of Sums

- This is Theorem A on page 217 of the textbook. Suppose c is any constant.

$a_1+b_1+a_2+b_2+\dots = a_1+a_2+a_3+\dots a_n+b_1+b_2+\dots b_n$

$$
\begin{array}{rrcl}
(i) & \sum_{i=1}^{n} c a_i & = & \sum_{i=1}^{n} a_i \\
(ii) & \sum_{i=1}^{n} (a_i + b_i) & = & \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \\
(iii) & \sum_{i=1}^{n} (a_i - b_i) & = & \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i
\end{array}
$$

$ca_1+ca_2+\dots = c(a_1+a_2+\dots)$

- These properties are sometimes summarized by saying the summation is **linear**.
- A little more subtle is the sometimes useful property

$a_1 + a_2 + \dots + a_m + a_{m+1} + \dots + a_n$

$$\sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i = \sum_{i=1}^{n} a_i$$

- A very special kind of sum that, however, occurs with reasonable frequency, is a **telescoping** or **collapsing sum**:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots$$

- Hint:

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1} = \frac{i+1-i}{i(i+1)}$$

$$= \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} \dots \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

Riemann Sums

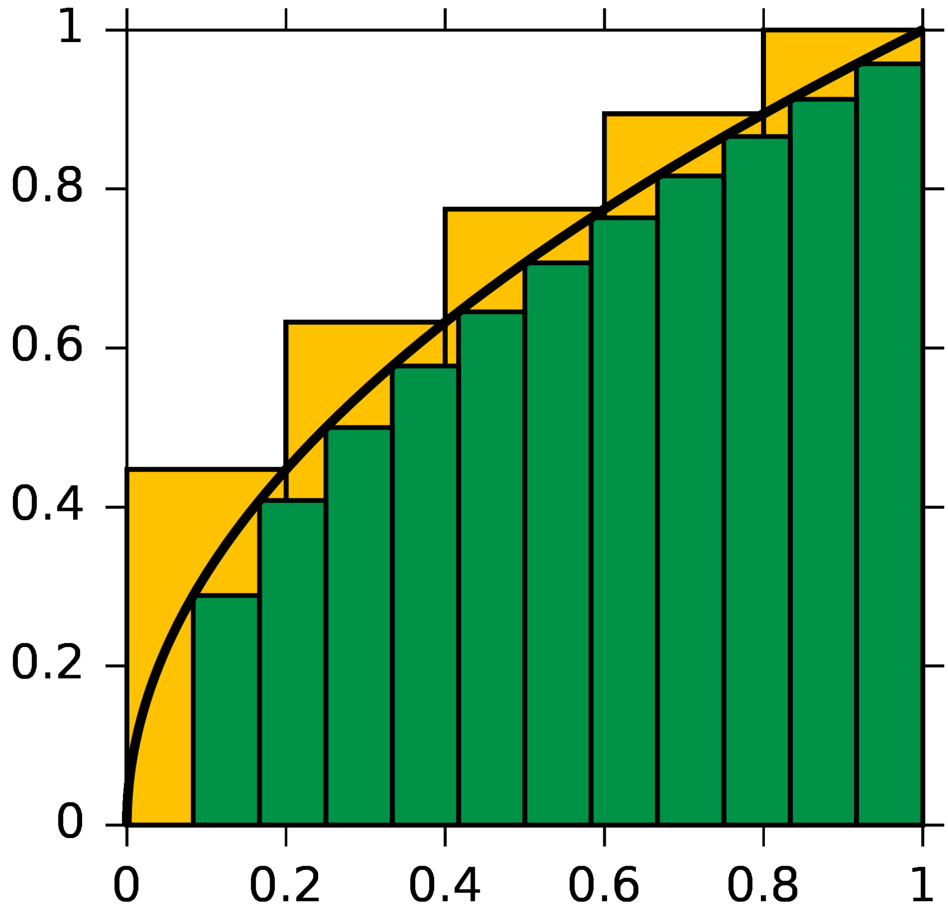


Figure 1. Riemann Sum.

- Recall the definition of the Riemann integral on page 226 of the textbook.

$$A = \int_a^b f(x)\mathrm{d}x = \lim_{\|P\| \longrightarrow 0} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i \qquad (1)$$

- The expression $\int_a^b f(x)\mathrm{d}x$ is called the **definite integral** of f with respect to x from a to b.
- If f is positive this is the area underneath the curve.
- x is the **integration variable**
- a and b are the **lower** and **upper limits of integration**.
- P is a **partition** of the interval $[a, b]$

$$P: \qquad a = x_0 < x_1 < \ldots < x_n = b \tag{2}$$

$$\Delta x_i = x_i - x_{i-1}, \qquad i = 1, 2, \ldots, n \tag{3}$$

$$\|P\| = \max_{i=1,\ldots,n} \Delta x_i \tag{4}$$

$\|P\|$ is called the **norm** of P.

$\bar{x}_i$ is any point in the interval $[x_{i-1}, x_i]$.

- Crucially, **the limit must be independent of how the partitions and the $\bar{x}_i$ are chosen**.
- For a given function f, the limit may not exist. If it does exist the function is **integrable**, otherwise it's **non-integrable**.

- What functions are integrable is a very subtle question, and a big issue in pure mathematics. For our purposes, which are focused on applications, it's enough to know that any function that is continuous, or piecewise continuous, is also integrable.
- Recall some special formulas (textbook, p. 218):

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

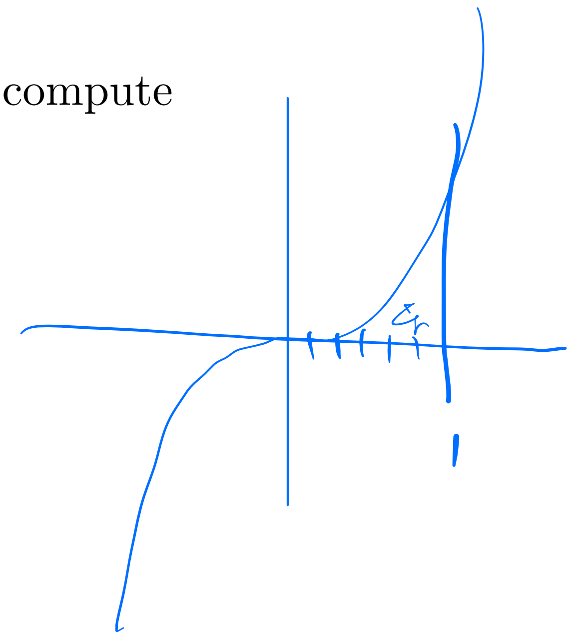
$$(n^2+1)^2 = n^4 + 2n^2 + 1$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{(n^2+1)^2}{4} = \frac{n^4 + LOT}{4}$$

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

- Example: Using the definition compute

$$I = \int_0^1 x^3 \mathrm{d}x$$



n subint.

$$\Delta x = \frac{1}{n}$$

$$x_i = \frac{i}{n} \qquad i = 0, \ldots, n$$

$$0 = x_0 = 0$$

$$x_1 = \frac{1}{n}$$

$$x_2 = \frac{2}{n}$$

$$x_3 = \frac{3}{n}$$

$$I \approx \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \frac{i^3}{n^3} \cdot \frac{1}{n}$$

$$= \frac{1}{n^4} \sum_{i=1}^{n} i^3 = \frac{1}{n^4} \frac{n^4 + LOT}{4}$$

$$I = \lim_{n \to \infty} \frac{n^4 + 4n^2 + 1}{4n^4} = \lim_{n \to \infty} \frac{n^4 + LOT}{4n^4} = \frac{1}{4}$$

FToC: $f(x) = x+3 = F'(x)$

$F(x) = \frac{x^2}{2} + 3x$

$F(3) = \frac{9}{2} + 9 = \frac{27}{2}$

$F(-2) = 2 + (-6) = -4$

- Example 3, p. 228, textbook.

$F(3) - F(-2) = \frac{27}{2} - (-4)$

$= \frac{27}{2} + 4 = \frac{35}{2}$

- Compute

$$I = \int_{-2}^{3} x + 3\mathrm{d}x.$$

$\Delta x = \frac{5}{n} \quad x_i = -2 + i\frac{5}{n}$

$[a, b]$

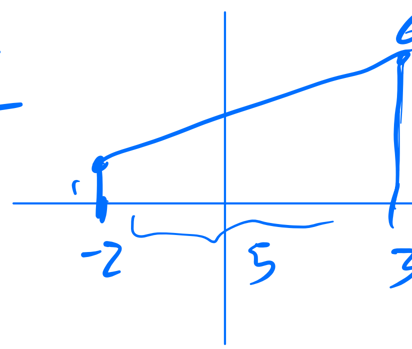
$\Delta x = \frac{b-a}{n}$

$I \approx \sum_{i=1}^{n} f(x_i)\Delta x$

$= \sum_{i=1}^{n} \left(-2 + \frac{5i}{n} + 3\right)\frac{5}{n} = \sum_{i=1}^{n}\left(1 + \frac{5i}{n}\right)\frac{5}{n}$

$f(x) = x+3$

$= \underbrace{\frac{5}{n}\sum_{i=1}^{n} 1}_{5} + \frac{5}{n}\sum_{i=1}^{n}\frac{5i}{n}$



$A = 5 \cdot \frac{7}{2}$

$= \frac{35}{2}$

$= 5 + \frac{5^2}{n^2}\sum_{i=1}^{n} i$

$= 5 + \frac{5^2}{n^2}\frac{n(n+1)}{2}$

$\lim_{n\to\infty} \frac{n+1}{n} = 1$

$= 5 + \frac{5^2(n+1)}{2n}$

$I = \lim_{n\to\infty}\left(5 + \frac{5^2(n+1)}{2n}\right) = 5 + \frac{25}{2} = \frac{35}{2}$

- Example 4, page 228, textbook.
- Example: Compute

$$I = \int_0^2 x^2 - 1 \mathrm{d}x.$$

exercise

(This is a simplification of Example 4 in the textbook.)

- In our standard procedure

$$\text{vague notion} \longrightarrow \text{definition} \longrightarrow \text{properties} \tag{5}$$

  the above definition is the middle stage.

- Very soon we will see that

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a) \quad \text{where} \quad F' = f \tag{6}$$

- So why, you ask, should we bother with the definition on the previous page?
- The value of that concept is the ability to go the other way: approximate something by a sum, recognize the limit of that sum as an integral, and then compute the integral.
- We will use that approach several times when we talk about applications for the rest of the semester.
- For that opposite approach we do not need to bother with intervals of differing lengths, and general evaluation points.
- But we do want to be able to interpret Riemann Sums as definite integrals.

- Another Example:
- Recall that

$$\int_a^b f(x)\mathrm{d}x = \lim_{\Delta x \longrightarrow 0} \sum_{i=1}^{n} f(x_i)\Delta x. \qquad (7)$$

- Suppose

$$\Delta x = \frac{\pi}{n}, \qquad x_i = i\Delta x \quad \text{and}$$

$$\sum_{i=1}^{n} \sin x_i \Delta x = \int_a^b f(x)\mathrm{d}x.$$

- Then

exercise

$$a =$$

$$b =$$

$$f(x) =$$

- Can you compute $\int_a^b f(x)\mathrm{d}x$?

Properties of the Definite Integral

- With this observation, the following properties of the definite integral follow straight from the
- Definition:

$$\int_a^b f(x)\mathrm{d}x = \lim_{\Delta x \longrightarrow 0} \sum_{i=1}^{n} f(x_i)\Delta x. \tag{8}$$

- Zero-Length interval:

$$\int_a^a f(x)\mathrm{d}x = 0 \tag{9}$$

- Just like in a sum the choice of the summation index does not matter, in an integral the choice of the integration variable does not matter. Of course, you would not want to use the same variable as you use for the upper or lower limit of integration.

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(z)\mathrm{d}z = \int_a^b f(t)\mathrm{d}t \tag{10}$$

- just like a sum can be split:

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} a_i \tag{11}$$

a definite integral can be split:

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x \tag{12}$$

- Integration is linear:

$\int_a^b f(t) + g(t)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$

$$\int_a^b \alpha f(t) + \beta g(t)\mathrm{d}t = \alpha \int_a^b f(t)\mathrm{d}t + \beta \int_a^b g(t)\mathrm{d}t \tag{13}$$

- Our definition can be generalized by dropping the requirement that $a < b$. The norm of the partition is then the maximum length of a subinterval. Correspondingly, Δx_i may be zero, positive, or negative. As an exercise, think about the details. This observation gives the following property

$$\int_a^b f(t)\mathrm{d}t = -\int_b^a f(t)\mathrm{d}t \tag{14}$$

$\Delta x = \frac{b-a}{n}$ $\Delta x = \frac{a-b}{n}$

- In other words, switching the limits of integration changes the sign of the definite integral

4.3-4.4 The Fundamental Theorem of Calculus

- We'll spend two days on sections 4.3 and 4.4 combined.
- Recall that we introduced derivatives and integrals by going back and forth between velocity and location.
- Naturally these two processes are inverses of each other.
- The Fundamental Theorem of Calculus (FToC) makes this precise.
- It comes in two flavors.
- Theorem A, page 235

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x)}$$

- Theorem A, page 243

$$\boxed{\int_a^b f(x)\mathrm{d}x = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x).}$$

- The textbook calls these the first and second fundamental theorem of Calculus, but the two statements are equivalent and there is really only one FToC.
- We need to learn how to use these facts, and we need to see why they are true and why they are equivalent.
- Before thinking about why the statements are true and equivalent, let's do some examples.
- Example:

  $$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x t^2 \mathrm{d}t =$$

- Example

  $$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^2 x^2 \mathrm{d}x =$$

- Example:

  $$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\pi \sin t \mathrm{d}t =$$

- Example:

  $$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^3 3x + 4 \mathrm{d}x =$$

- Example:

  $$\frac{\mathrm{d}}{\mathrm{d}x}\int_3^5 4x^3 + 1 \mathrm{d}x =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} f(t)\mathrm{d}t =$$