Math 1210-23

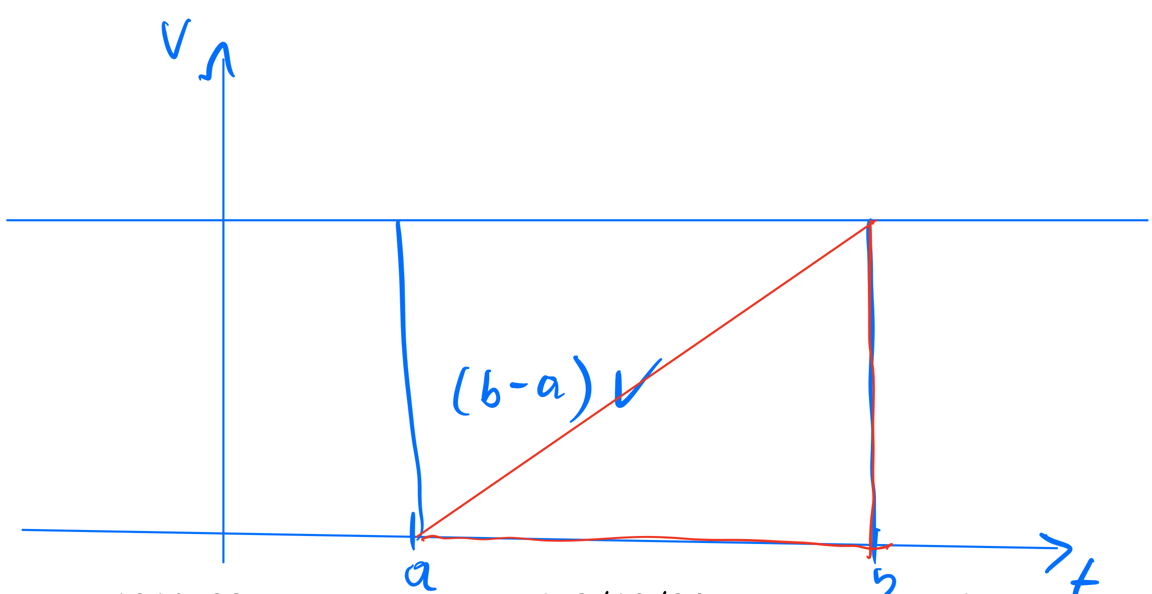
Notes of 3/19/24

4.1 Introduction to Area

- Recall the "indefinite integral" or "antiderivative":

$$\int f(x)\mathrm{d}x = F(x) + C \qquad \text{where} \qquad F' = f.$$

- There is also a closely related **definite integral**.
- It gives the area underneath a curve.
- Recall that at the beginning of this semester we computed velocity by interpreting it as the area underneath a curve.



Example: Let's compute the area of the region enclosed by the x-axis, the line $x = 1$, the y-axis, and the graph of $y = f(x) = x^2$.

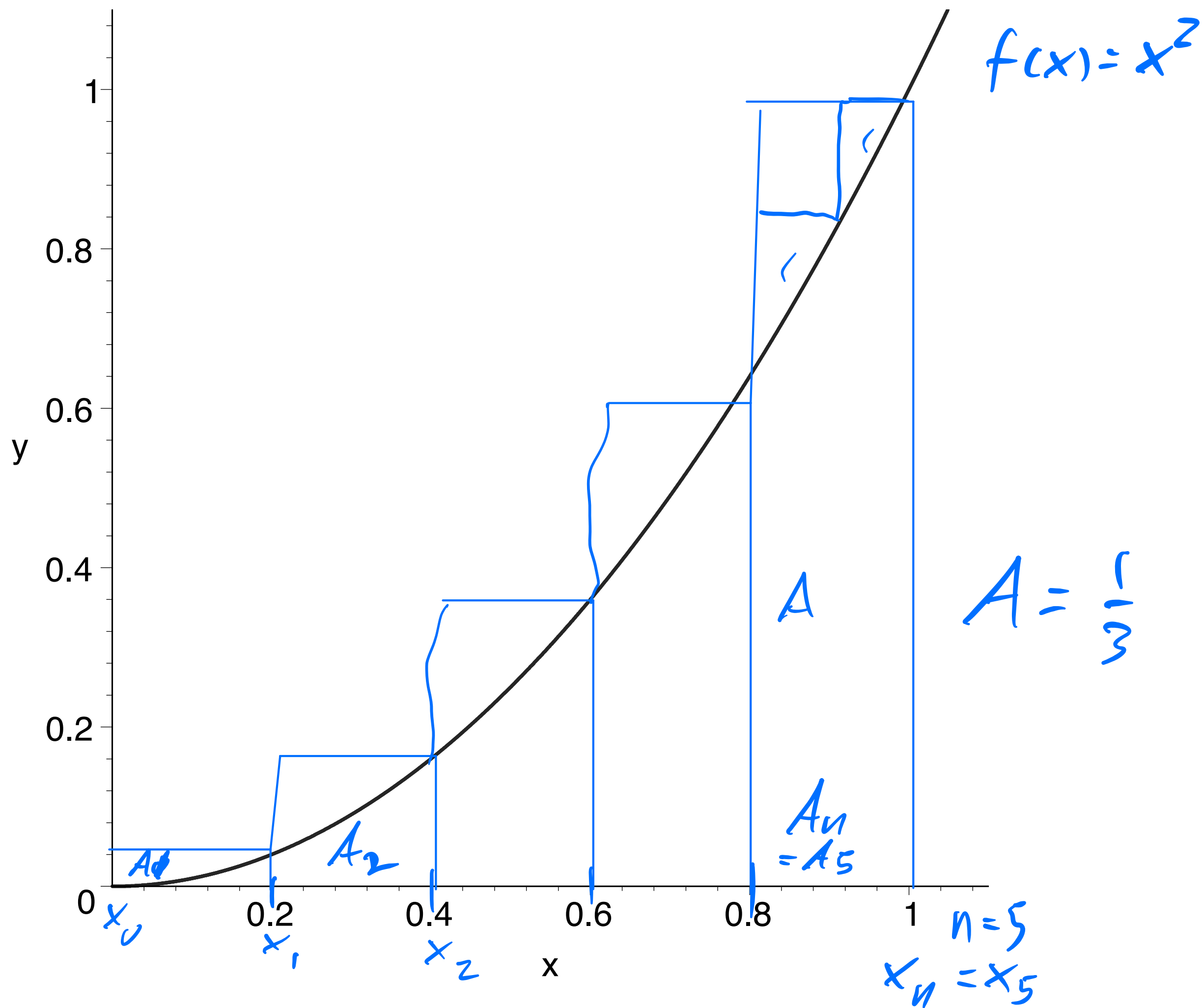


Figure 1. $f(x) = x^2$.

- Useful Hint:

$$1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$$

$$x_i = \frac{i}{n} \qquad i = 0, \dots, n$$

$$A_i = \frac{1}{n} \cdot x_i^2 \qquad i = 1, \dots, n$$

$$= \frac{1}{n} \cdot \left(\frac{i}{n}\right)^2 = \frac{1}{n} \frac{i^2}{n^2} = \frac{i^2}{n^3}$$

$$A \approx A_1 + A_2 + \dots + A_n$$

$$= \frac{1^2}{n^3} + \frac{2^2}{n^3} + \dots + \frac{n^2}{n^3}$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + \dots + n^2\right)$$

$$= \frac{2n^3 + 3n^2 + n}{6n^3}$$

$$A = \lim_{n \to \infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{2}{6} = \frac{1}{3}$$

- We'll study this and related concepts for the remainder of the semester..
- Basic idea: Approximate area by the area of a staircase, take the limit as the number of stairs goes to infinity, and their widths go to zero.

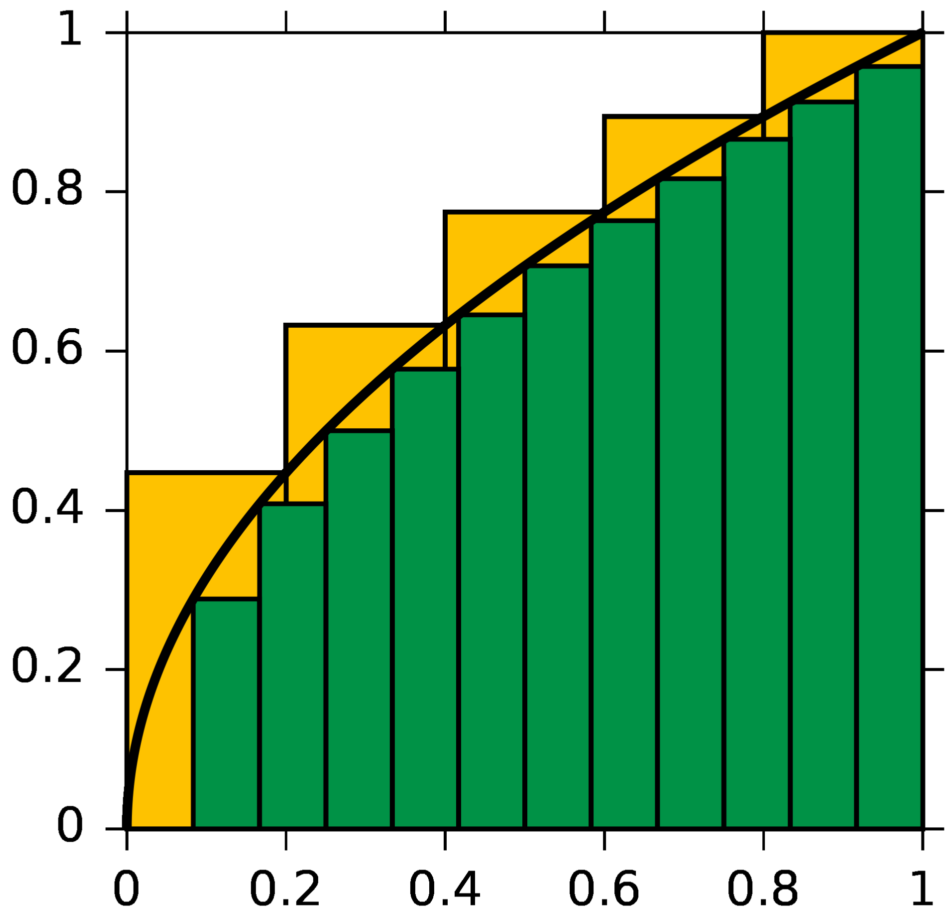


Figure 2. Riemann Sum.

- From `http://kkleemaths.com/wp-content/uploads/2016/07/2000px-Integral_approximations.svg_.png`

Note that we can interpret this picture also in terms of velocity and location. If the velocity is constant then the distance covered in a time interval Δt equals the product of velocity and Δt, i.e., the area of the associated rectangle.

Sum Notation

- Quick Review of **Sum** or **Sigma notation**

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n = \sum_{j=1}^{n} a_j$$

 introduced in College Algebra

- i or j, the **summation index** has no meaning outsider the sum.
- Σ is the capital Greek letter Sigma.
- Here are some useful special formulas (textbook, p. 218):

$$\sum_{i=1}^{n} 1 = n$$

$a_i = 1$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$a_i = i$

$\underline{5050}$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

- Thus we approximate the area underneath a graph by the area of a staircase. Then we consider what happens as the number of stairs goes to infinity and the width of the stairs goes to zero.

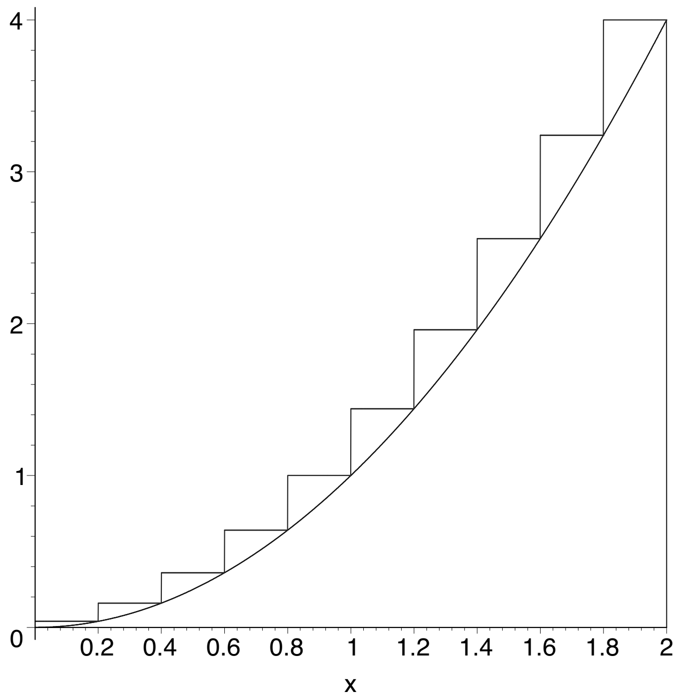


Figure 3. Areas.

$$\Delta x = \frac{b-a}{n}, \qquad x_i = a + i\Delta x \tag{1}$$

- the area of the box from x_{i-1} to x_i is $f(x_i)\Delta x$ and the total area is

$$\boxed{A = \lim_{\Delta x \longrightarrow 0} \sum_{i=1}^{n} f(x_i)\Delta x}. \tag{2}$$

- The expression $\sum_{i=1}^{n} f(x_i)\Delta x$ is an example of a **Riemann Sum**.
- Bernhard Riemann, 1826–1866.
- To get the standard definition of the Riemann Integral we need to get a little more general in the following ways:
  - allow for the subintervals to have different lengths
  - allow for evaluation of f at points other than the left endpoint of those subintervals
  - take the limit as the maximum size of those intervals goes to zero.
- This gives rise to the celebrated definition on page 226 of our textbook.

$$A = \int_a^b f(x)\mathrm{d}x = \lim_{\|P\| \longrightarrow 0} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i \qquad (3)$$

- The expression $\int_a^b f(x)\mathrm{d}x$ is called the **definite integral** of f with respect to x from a to b.
- If f is positive this is the area underneath the curve.
- x is the **integration variable**
- a and b are the **lower** and **upper limits of integration**.

- P is a **partition** of the interval $[a, b]$

$$P: \qquad a = x_0 < x_1 < \ldots < x_n = b \tag{4}$$

$$\Delta x_i = x_i - x_{i-1}, \qquad i = 1, 2, \ldots, n \tag{5}$$

$$\|P\| = \max_{i=1,\ldots,n} \Delta x_i \tag{6}$$

  $\|P\|$ is called the **norm** of P.

  $\bar{x}_i$ is any point in the interval $[x_{i-1}, x_i]$.

- Crucially, **the limit must be independent of how the partitions and the $\bar{x}_i$ are chosen**.

- For a given function f, the limit may not exist. If it does exist the function is **integrable**, otherwise it's **non-integrable**.

- What functions are integrable is a very subtle question, and a big issue in pure mathematics. For our purposes, which are focused on applications, it's enough to know that any function that is continuous, or piecewise continuous, is also integrable.

- In our standard procedure

$$\text{vague notion} \longrightarrow \text{definition} \longrightarrow \text{properties} \tag{7}$$

the above definition is the middle stage.

- Very soon we will see that

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a) \quad \text{where} \quad F' = f \tag{8}$$

- So why, you ask, should we bother with the definition on the previous page?
- The value of that concept is the ability to go the other way: approximate something by a sum, recognize the limit of that sum as an integral, and then compute the integral.
- We will use that approach several times when we talk about applications for the rest of the semester.
- For that opposite approach we do not need to bother with intervals of differing lengths, and general evaluation points.
- But we do want to be able to interpret Riemann Sums as definite integrals.

- Recall

$$\Delta x = \frac{b-a}{n}, \qquad x_i = a + i\Delta x \tag{9}$$

$i = 0, \dots, n$

- $$\int_a^b f(x)\mathrm{d}x = \lim_{\Delta x \longrightarrow 0} \sum_{i=1}^{n} f(x_{i\cancel{-1}})\Delta x. \tag{10}$$

- You want to be able to read this equation from right to left.

Examples

- Example 0:

$$\Delta x = \frac{1}{n}, \qquad x_i = i\Delta x \tag{11}$$

$$\lim_{n \longrightarrow \infty} \sum_{i=1}^{n} x_i^2 \Delta x = \int_a^b f(x)\mathrm{d}x \tag{12}$$

where

$$a = 0$$

$$F(x) = \frac{x^3}{3} + C$$

$$b = 1 \tag{13}$$

$$f(x) = x^2$$

$$\int_0^1 x^2 dx = F(1) - F(0) = \frac{1^3}{3} + C - \left(\frac{0^3}{3} + C\right)$$

$$= \frac{1}{3}$$

- Example 1:

$$\Delta x = \frac{3}{n}, \qquad x_i = i\Delta x \tag{14}$$

$$\lim_{n\longrightarrow\infty} \sum_{i=1}^{n} x_i^2 \Delta x = \int_a^b f(x)\mathrm{d}x \tag{15}$$

where

$$\int_0^3 x^2 dx$$

$$x_0 = a = 0$$
$$x_n = b = 3 \tag{16}$$
$$f(x) = x^2$$

$$\int_0^3 x^2 dx = \frac{3^3}{3} - \frac{0^3}{3} = 9$$

- Example 2:

$$\Delta x = \frac{3}{n}, \qquad x_i = 2 + i\Delta x \tag{17}$$

$$\lim_{n \longrightarrow \infty} \sum_{i=1}^{n} x_i^2 \Delta x = \int_a^b f(x) \mathrm{d}x \tag{18}$$

where

$$\begin{aligned} a &= 2 \\ b &= 5 \\ f(x) &= x^2 \end{aligned} \tag{19}$$

$$\int_2^5 x^2 dx = \frac{5^3}{3} - \frac{2^3}{3} = \text{WTI}$$

- Example 3:

$$\Delta x = \frac{3}{n}, \qquad x_i = -1 + i\Delta x \tag{20}$$

$$\lim_{n\longrightarrow\infty} \sum_{i=1}^{n} g^2(x_i)\Delta x = \int_a^b f(x)\mathrm{d}x \tag{21}$$

$$\int_{-1}^{2} g^2(x)\,dx$$

where

$$\begin{aligned} a &= -1 \\ b &= 2 \\ f(x) &= g^2(x) \end{aligned} \tag{22}$$

- Example 4:

$$\Delta x = \frac{2}{n}, \qquad x_i = -1 + i\Delta x \tag{23}$$

$$\lim_{n \longrightarrow \infty} \sum_{i=1}^{n} \sqrt{1 - x_i^2}\,\Delta x = \int_a^b f(x)\mathrm{d}x \tag{24}$$

where

$$x^2 + y^2 = r^2 = 1^2$$

$$y^2 = r^2 - x^2$$

$$y = \sqrt{r^2 - x^2}$$

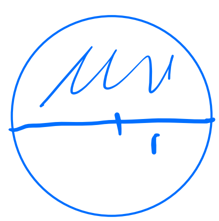
$$= \sqrt{1 - x^2}$$

$$a = -1$$

$$b = 1$$

$$f(x) = \sqrt{1 - x^2} \tag{25}$$

$$\int_a^b f(x)\mathrm{d}x = \int_{-1}^{1} \sqrt{1 - x^2}\,dx = \frac{\pi}{2}$$



Properties

- Our definition can be generalized by dropping the requirement that $a < b$. The norm of the partition is then the maximum length of a subinterval. Correspondingly, Δx_i may be zero, positive, or negative. As an exercise, think about the details.
- With this observation, the following properties of the definite integral follow straight from the
- Definition:

$$\int_a^b f(x)\mathrm{d}x = \lim_{\Delta x \longrightarrow 0} \sum_{i=1}^n f(x_i \Delta x. \tag{26}$$

- Zero-Length interval:

$$\int_a^a f(x)\mathrm{d}x = 0 \tag{27}$$

- Just like in a sum the choice of the summation index does not matter, in an integral the choice of the integration variable does not matter. Of course, you would not want to use the same variable as you use for the upper or lower limit of integration.

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(z)\mathrm{d}z = \int_a^b f(t)\mathrm{d}t \tag{28}$$

- just like a sum can be split:

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} a_i \tag{29}$$

a definite integral can be split:

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x \tag{30}$$

- Integration is linear:

$$\int_a^b \alpha f(t) + \beta g(t)\mathrm{d}t = \alpha \int_a^b f(t)\mathrm{d}t + \beta \int_a^b g(t)\mathrm{d}t \tag{31}$$

- Switching the limits of integration changes the sign of the definite integral

$$\int_a^b f(t)\mathrm{d}t = -\int_b^a f(t)\mathrm{d}t \tag{32}$$