3.7 Newton's Method

Math 1210-4

Notes of 10/27/2017

How to compute the square root of 2?

\[ f(x) = x^2 - 2 = 0 \]

\[ (2, 0) \]

\[ (x_0, f(x_0)) \]

\[ \bar{x} = \frac{x_0^2 - 2}{2x_0} \]

\[
\bar{y} = T(x)
\]

\[
\bar{x} = x_0 - \frac{f(x_0)}{f'(x_0)}
\]

\[
\bar{x} = x_0 - \frac{x_0^2 - 2}{2x_0}
\]
3.7 Newton’s Method

• In general, here is Newton’s Method for solving

$$f(x) = 0$$

\[
x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}
\]
Figure 1. Newton’s Method.

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]
• For $f(x) = x^2 - 2$ we get:

```maple
Digits := 50:
f := x^2 - 2:
g := x - f/diff(f, x):
xn := 1:
lprint(sqrt(2.0)):
xn := evalf(subs(x = xn, g)):
lprint(i, xn):
end do:
```

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• Same for \( f(x) = x^3 - 2 \), computing the cube root of 2.

```maple
restart:Digits:=50:
f:=x^3-2:
g:=x-f/diff(f,x):
xn:=1:
lprint(evalf(2.0**(1/3))):
1.2599210498948731647672106072782283505702514647015
>
for i from 1 to 8 do
  xn:=evalf(subs(x=xn,g)):
lprint(i,xn):
end do:
1, 1.33333333333333333333333333333333333333
2, 1.263888888888888888888888888888888888888889
3, 1.2599334934499769664604829439994275159110323945489
4, 1.2599210500177697737293010979898432536537097958626
5, 1.2599210498948731647791983238845005280761096256985
6, 1.2599210498948731647672106072782283505703655237129
7, 1.2599210498948731647672106072782283505702514647015
8, 1.2599210498948731647672106072782283505702514647015
>
quit
```
• Here is a different view of Newton’s Method. It’s given

\[ x_{n+1} = g(x_n) \quad \text{where} \quad g(x) = x - \frac{f(x)}{f'(x)}. \]

• Clearly

\[ f(x) = 0 \iff x = g(x). \]

• \( x \) is a **fixed point** of \( g \) if \( x = g(x) \). Finding \( x \) such that \( x = g(x) \) is a **Fixed Point Problem**.

• Defining a sequence by

\[ x_{n+1} = g(x_n) \]

is a **fixed point iteration**.

• Newton’s Method is a special case of a fixed point iteration.

• (Many) others are possible, for example

\[ g(x) = x + f(x) \quad \text{or} \quad g(x) = x - f(x). \]

• The following five drawings illustrate different possible behaviors of fixed point iterations.
Figure 2. Fixed Point Iteration 1.

$x_0 = 0.2$
$x_1 = g(x_0)$

Figure 3. Fixed Point Iteration 2.
Figure 4. Fixed Point Iteration 3.

Figure 5. Fixed Point Iteration 4.
Figure 6. $g(x) = 4x(1 - x)$.

- This is an instance of what in mathematics is called **chaos**. If you are interested you can take Math 5470, Chaos and Nonlinear Systems, for more info.
• What is going on?

\[ f(z) = 0 \quad z = g(z) \]

\[ z - x_{n+1} = g(z) - g(x_n) \]

\[ = g'(c)(z - x_n) \]

\[ g(x) = x - \frac{f(x)}{f'(x)} \]

\[ g'(x) = 1 - \frac{f'(x)}{f''(x)} \]

\[ = 1 - \frac{f'(x)}{f'(x)} + \frac{f(x)f''(x)}{f'(x)^2} \]

\[ = \frac{f(x)f''(x)}{f'(x)^2} \quad |g'(z)| < 1 \]

\[ g'(z) = \frac{f(z)f''(z)}{f'(z)^2} = 0 \]
• Why does Newton’s Method work so well?
• How do you start?
• It depends on the problem!
• You need to have a good idea of where to look for the solution of your problem.
• For example: Find the roots of

\[ f(x) = 10(x - 1)(x - 2)(x - 3) - 1 \]

• This is problem 17 on hw 10 (which is now open).
• Commercial: Study Numerical Analysis

• Math 5600, 5610, 5620, 6610, 6620

• Prerequisite: solid Calculus (1210-1220-2210), linear algebra (matrices and vectors) (Math 2270), and some programming experience (the more the better).
\[ V = \pi r^2 h \]
\[ r^2 + \frac{h^2}{4} = R^2 \]
\[ r^2 = R^2 - \frac{h^2}{4} \]

\[
= \pi \left( R^2 - \frac{h^2}{4} \right) h = f(h) \\
= \pi \left( R^2 h - \frac{h^3}{4} \right) = f(h) \\
f'(h) = \pi \left( R^2 - \frac{3h^2}{4} \right) = 0 \\
\]
\[ h^2 = \frac{4}{3} R^2 \]
\[ h = \frac{2}{\sqrt[3]{R}} \]

\[ I = \frac{I_1}{x^2} + \frac{I_2}{(d-x)^2} = \text{min} \]

\[ I_1 \]

\[ I_2 \]