Math 1210-23

Notes of 3/18/24

- Recall from Friday:
- F is an *antiderivative* of f if

$$F' = f$$

Notation

- There are many notations for derivatives but essentially only one notation (due to Leibniz) for antiderivatives:

$$F(x) = \int f(x)\mathrm{d}x$$

- $\int f(x)\mathrm{d}x$ is the **set of all antiderivatives of f with respect to x.**
- Depending on the context it may also stand for a specific antiderivative.
- $f(x)$ is the **integrand**.
- x is the **integration variable**.
- $\int f(x)\mathrm{d}x$ is pronounced "integral f of x dee-x"
- $\int f(x)\mathrm{d}x$ is also called the **indefinite integral** (of f with respect to x).

- Basic idea of integration by substitution
- It's the inverse process of the chain rule.
- We know that

$$I = \int f\big(g(x)\big)g'(x)\mathrm{d}x = F\big(g(x)\big) + C$$

  where $F' = f$.

- Setting $u = g(x)$ we get

$$\frac{\mathrm{d}u}{\mathrm{d}x} = g'(x) \quad \text{and} \quad \mathrm{d}u = g'(x)\mathrm{d}x.$$

  The integral becomes

$$I = \int f\big(g(x)\big)g'(x)\mathrm{d}x = \int f(u)\mathrm{d}u = F(u) + C = F\big(g(x)\big) + C$$

- We'll use integration by substitution in many different ways on many occasions, but in the mean time here are a couple of simple examples:

- Example 5, textbook: Evaluate

$$\int (x^4 + 3x)^{50}(4x^3 + 3)\mathrm{d}x$$

- Example 5 continued: Evaluate

$$\int \sin^{10} x \cos x \mathrm{d}x$$

- Evaluate $\int \sin x \cos x \mathrm{d}x$ in two different ways.

3.9 Introduction to Differential Equations

- Not really an introduction. Rather this is the first glimpse of a very large iceberg.
- DE = "Differential Equation"
- U of U Math courses on DEs:

2250	2280	3140	3150	5410	5420	5440
5470	5500	5620	6410	6420	6430	6440
6620	6630	6750	6840	6845	6850	6865

- All of those courses are at least half focused on DEs, most are devoted completely to DEs.
- DEs are important because they can be used to model natural processes.
- Ask professors of your major about the importance of DEs.
- So, what's a differential equation?

- A differential equation is an equation that involves a function and some of its derivatives.

Examples

1. Antiderivatives: Solutions of

$$F'(x) = f(x)$$

We defined last week

$$F(x) = \int f(x)\mathrm{d}x$$

- (This is the reason why DEs occur at this point in the curriculum. What we did last Friday is a special case of what we are doing today.)

2. Falling Object:

$$h''(t) = v'(t) = a(t) = -32, \qquad h(0) = h_0, \quad h'(0) = v_0$$

$$h'(t) = v(t) = -32t + v_0$$

$$h(t) = -16t^2 + v_0 t + h_0$$

3. Example:

$$s'' = -s$$

4. Example:

$$s' = -s^2$$

5. Example 2, textbook

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + 3x^2}{y^2}$$

- This is a "separable DE". To solve it separate variables and then integrate on both sides

6. Example 5 in the textbook is optional but hugely interesting.

- The gravitational force F exerted by the Earth on an object of mass m at a distance s from the center of the Earth is inversely proportional to the square of s. How far will an object fly that is launched straight up with an initial velocity v_0?
- In other words, with R being the radius of Earth, and g being the acceleration due to gravity near the surface of the earth,

$$F = -\frac{mgR^2}{s^2}$$

- According to Newton's second law, force equals mass times acceleration. This gives

$$F = m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}s}v = -mg\frac{R^2}{s^2}.$$

- The last equation is a separable DE in

$$v = v(s),$$

  the velocity as a function of distance.

- Dividing by m, and separating variables gives

$$v\mathrm{d}v = -g\frac{R^2}{s^2}\mathrm{d}s$$

- Integrating on both sides gives

$$\frac{1}{2}v^2 = \frac{gR^2}{s} + C \tag{1}$$

- We determine the integration constant by observing that $v = v_0$ when $s = R$. Substituting in (1) gives

$$\frac{1}{2}v_0^2 = \frac{gR^2}{R} + C = gR + C$$

- Thus
$$C = \frac{1}{2}v_0^2 - gR$$

- Substituting in (1) and multiplying with 2 gives
$$v^2 = \frac{2gR^2}{s} + v_0^2 - 2gR$$

- We reach the maximum height when v^2 becomes zero, i.e.,

$$v^2 = \frac{2gR^2}{s} + v_0^2 - 2gR = 0 \tag{2}$$

- We can easily solve this equation for

$$s = \frac{2gR^2}{2gR - v_0^2}$$

However, note that v^2 in (2) remains positive for all distances if

$$\boxed{v_0^2 > 2gR} \tag{3}$$

- In that case the object never returns to Earth!
- (Our solution for s in that case is negative which does not make physical sense.)

- The critical velocity
$$v_0 = \sqrt{2gR}$$
is the **Escape Velocity**.
- An object leaving the surface of Earth at that velocity will never return, but its velocity will approach zero.
- An object leaving the surface of Earth at a larger velocity will also never return, of course, but its velocity will approach a positive value as the distance from Earth increases.
- Simplifying things by assuming the Earth is a sphere with a radius of 6,371 km (3,959 miles) we get an escape velocity on the surface of Earth of 11.2 km/second, about 6.97 miles per second, or Mach 33.
- Here is a table of some other escape velocities, taken from the wikipedia (slightly modified). We assume we are at initially at rest at the given location. (Escaping from the sun starting on the orbiting Earth is more complicated.)

Location	with respect to	v_0(km/sec)
Earth	Earth's Gravity	11.2
Moon	Moon's Gravity	2.38
Sun surface	Sun's Gravity	617.5
Earth	Sun's Gravity	42.1
Solar System	Milky Way	≈ 500
Event Horizon	Black Hole	Speed of Light

- Start with Areas.