study session after class!

Math 1210-23

Notes of 3/15/24

3.8 Antiderivatives

- Very simple idea: instead of differentiating, go the other way.
- Example: Go from acceleration to velocity to location.

h(t) height of the rock
initial speed = 6 f/s
release at height of 5'

$$v'(t) = a(t) = -32 \ ft/s^2$$
$$h'(t) = v(t) = -32t + 6$$
$$h(t) = -16t^2 + 6t + 5$$

$$a(t) = -g$$
$$v(t) = -gt + v_0$$
$$h(t) = -\frac{gt^2}{2} + v_0 t + h_0$$

- F is an **antiderivative** of f if

$$F' = f,$$

  i.e., f is the derivative of F.

- Example: $f(x) = 2x$.

$$F(x) = x^2 + C$$

- Antiderivatives are determined only up to a constant.

- Contrast this with derivatives. Given a function F, its derivative F', if it exists, is **unique**.

- That constant is called the **integration constant** and is often denoted by C.

- Computation of an antiderivative is called **integration**.

- We can differentiate or integrate a function.

- These processes are inverses. If we first integrate and then differentiate we are back at where we started.

$$2x \xrightarrow[I]{} x^2 \xrightarrow[D]{} 2x$$

- What if we first differentiate and then integrate?

$$x^2 \xrightarrow[D]{} 2x \rightarrow x^2 + C$$

More Examples

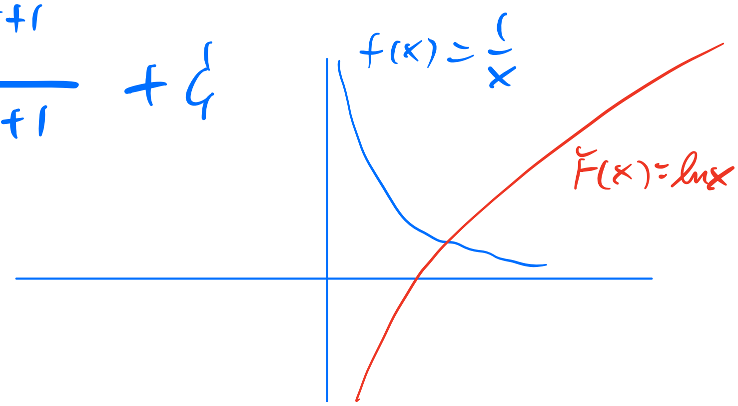
$$f(x) = 0 \qquad F(x) = C$$

$$f(x) = 1 \qquad F(x) = x + C$$

$$f(x) = x^2 \qquad F(x) = \frac{x^3}{3} + C \qquad F'(x) = x^2 + 0 = x^2$$

$$f(x) = x^3 \qquad F(x) = \frac{x^4}{4} + C$$

$$f(x) = x^r \qquad F(x) = \frac{x^{r+1}}{r+1} + C$$



$$x^{-1} = \frac{1}{x}$$

What if $r = -1$?

$$f(x) = x^{-1} \qquad F(x) =$$

Notation

- The notation is due to Gottfried Wilhelm Leibniz, one of two inventors of Calculus.

Figure 1. Gottfried Leibniz, 1646-1716 (wikipedia).

- There are many notations for derivatives but essentially only one notation (due to Leibniz) for antiderivatives:

$$F(x) = \int f(x) \mathrm{d}x$$

- $\int f(x)\mathrm{d}x$ is the **set of all antiderivatives of f with respect to x.**

- Depending on the context it may also stand for a specific antiderivative.

- The symbol $\int$ is the **integration symbol**. It has this particular shape since integrals can be defined as sums (as we will see) and the integration symbol can be thought of as an elongated letter S.

- $f(x)$ is the **integrand**.

- x is the **integration variable**.

- $\int f(x)\mathrm{d}x$ is pronounced "integral f of x dee-x"

- $\int f(x)\mathrm{d}x$ is also called the **indefinite integral** (of f with respect to x).

- **Aside:** If there is an indefinite integral there must be a definite one! Indeed, there is! In fact, definite integrals will be the focus of our attention for the remainder of the semester, starting next week.

- More examples:

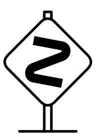
$$\int x^2 + 1\mathrm{d}x = \frac{x^3}{3} + C_1 + x + C_2 = \frac{x^3}{3} + x + C$$

$$\int \sin x\mathrm{d}x = -\cos x + C$$

$$\int \cos x\mathrm{d}x = \sin x + C$$

$$\int x\mathrm{d}x = \frac{x^2}{2} + C$$

$$\int k\,dx = kx + C$$
$$\int 3\,dx = 3x + C$$



$$\int x\mathrm{d}t = xt + C$$

$$\int kf(x)\mathrm{d}x = k\int f(x)\,dx$$

$$\int f(x) + g(x)\mathrm{d}x = \int f(x)\,dx + \int g(x)\,dx$$

$$\int 2x(x^2+1)^{100}\mathrm{d}x = \frac{(x^2+1)^{101}}{101} + C$$

$$\int (x^2+1)^{100}\mathrm{d}x = \text{tricky}$$

Differentiation and Integration as Operators

- Both Differentiation and Integration take a function and return a function. Both processes are functions whose dependent and independent variables are functions!
- Such functions are called **operators**
- Differentiation and Integration have special properties, in particular:

$$\frac{\mathrm{d}}{\mathrm{d}x} k f(x) = k \frac{\mathrm{d}}{\mathrm{d}x} f(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x) + g(x)) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \frac{\mathrm{d}}{\mathrm{d}x} g(x)$$

and

$$\int k f(x) \mathrm{d}x = k \int f(x) \mathrm{d}x$$

$$\int f(x) + g(x) \mathrm{d}x = \int f(x) \mathrm{d}x + \int g(x) \mathrm{d}x$$

- Operators with those properties are called **linear**
- In general, an operator L is called **linear** if, for all functions f and g in the domain of L and all constants k:

$$L(kf) = kLf \quad \text{and} \quad L(f+g) = Lf + Lg.$$

Generalized Power Rule

$$\int [g(x)]^r g'(x)\mathrm{d}x = \frac{[g(x)]^{r+1}}{r+1} + C$$

- Examples:

$$\int (\sin^2 x)\cos x \, dx = \frac{\sin^3 x}{3} + C$$

$$\int (x^2+1)^{100} \cdot 2x \, dx = \frac{(x^2+1)^{101}}{101} + C$$

$$\int (x^3+1)^{50} \cdot x^2 \, dx = \frac{1}{3}\frac{(x^3+1)^{51}}{51} + C$$

$$\int (x^3+1)^{50} \cdot x \, dx = \text{tricky}$$

Integration by Substitution

- Consider again

$$\int [g(x)]^r g'(x)\mathrm{d}x = \frac{[g(x)]^{r+1}}{r+1} + C$$

$$I = \int (x^2+1)^{100}\, 2x\,dx$$

$$u = (x^2+1)$$

$$\frac{du}{dx} = 2x$$

$$du = 2x\,dx$$

$$I = \int u^{100}\,du = \frac{u^{101}}{101} + C_1$$

$$= \frac{(x^2+1)^{101}}{101} + C_1$$

CBD

- More Examples for integration by substitution:

  $\int (x^2+4)^{10} x \mathrm{d}x =$

  $\int \sin^3 x \mathrm{d}x =$

Rules to Live By

- Always check your integration by differentiation
- Integration can be very tricky, but differentiation is easy!
- When integrating you can guess and waive your hands, but if you can verify your answer by differentiation it's the correct one!
- So once you have the answer, by hook or crook, check it by differentiation.

What Integration Constant?

- The integration constant is often determined by side conditions.
- Example:

$$f(x) = F'(x) = 4x^3, \qquad \underline{F(1) = 5.} \qquad F(x) = x^4 + 4$$

$$\int 4x^3 dx = x^4 + c_1 = F(x)$$

$$1 + c_1 = 5$$

$$c_1 = 4$$

More Examples