Math 1210-23

Notes of 3/15/24

3.8 Antiderivatives

- Very simple idea: instead of differentiating, go the other way.
- Example: Go from acceleration to velocity to location.

- F is an **antiderivative** of f if

$$F' = f,$$

  i.e., f is the derivative of F.

- Example: $f(x) = 2x$.

- Antiderivatives are determined only up to a constant.

- Contrast this with derivatives. Given a function F, its derivative F', if it exists, is **unique**.

- That constant is called the **integration constant** and is often denoted by C.

- Computation of an antiderivative is called **integration**.

- We can differentiate or integrate a function.

- These processes are inverses. If we first integrate and then differentiate we are back at where we started.

- What if we first differentiate and then integrate?

More Examples

$$f(x) = 0 \qquad F(x) =$$

$$f(x) = 1 \qquad F(x) =$$

$$f(x) = x^2 \qquad F(x) =$$

$$f(x) = x^3 \qquad F(x) =$$

$$f(x) = x^r \qquad F(x) =$$

What if $r = -1$?

$$f(x) = x^{-1} \qquad F(x) =$$

Notation

- The notation is due to Gottfried Wilhelm Leibniz, one of two inventors of Calculus.

Figure 1. Gottfried Leibniz, 1646-1716 (wikipedia).

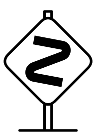
- There are many notations for derivatives but essentially only one notation (due to Leibniz) for antiderivatives:

$$F(x) = \int f(x) \mathrm{d}x$$

- $\int f(x)\mathrm{d}x$ is the **set of all antiderivatives of f with respect to x.**
- Depending on the context it may also stand for a specific antiderivative.
- The symbol $\int$ is the **integration symbol**. It has this particular shape since integrals can be defined as sums (as we will see) and the integration symbol can be thought of as an elongated letter S.
- $f(x)$ is the **integrand**.
- x is the **integration variable**.
- $\int f(x)\mathrm{d}x$ is pronounced "integral f of x dee-x"
- $\int f(x)\mathrm{d}x$ is also called the **indefinite integral** (of f with respect to x).

- **Aside:** If there is an indefinite integral there must be a definite one! Indeed, there is! In fact, definite integrals will be the focus of our attention for the remainder of the semester, starting next week.

- More examples:



$$
\begin{aligned}
\int x^2 + 1 \mathrm{d}x &= \\
\int \sin x \mathrm{d}x &= \\
\int \cos x \mathrm{d}x &= \\
\int x \mathrm{d}x &= \\
\int x \mathrm{d}t &= \\
\int k f(x) \mathrm{d}x &= \\
\int f(x) + g(x) \mathrm{d}x &= \\
\int 2x(x^2+1)^{100} \mathrm{d}x &= \\
\int (x^2+1)^{100} \mathrm{d}x &=
\end{aligned}
$$

Differentiation and Integration as Operators

- Both Differentiation and Integration take a function and return a function. Both processes are functions whose dependent and independent variables are functions!
- Such functions are called **operators**
- Differentiation and Integration have special properties, in particular:

$$\frac{\mathrm{d}}{\mathrm{d}x} k f(x) = k \frac{\mathrm{d}}{\mathrm{d}x} f(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x) + g(x)) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \frac{\mathrm{d}}{\mathrm{d}x} g(x)$$

and

$$\int k f(x) \mathrm{d}x = k \int f(x) \mathrm{d}x$$
$$\int f(x) + g(x) \mathrm{d}x = \int f(x) \mathrm{d}x + \int g(x) \mathrm{d}x$$

- Operators with those properties are called **linear**
- In general, an operator L is called **linear** if, for all functions f and g in the domain of L and all constants k:

$$L(kf) = kLf \quad \text{and} \quad L(f+g) = Lf + Lg.$$

Generalized Power Rule

$$\int [g(x)]^r g'(x) \mathrm{d}x = \frac{[g(x)]^{r+1}}{r+1} + C$$

- Examples:

Integration by Substitution

- Consider again

$$\int [g(x)]^r g'(x)\mathrm{d}x = \frac{[g(x)]^{r+1}}{r+1} + C$$

- More Examples for integration by substitution:

  $\int (x^2+4)^{10} x \mathrm{d}x =$

  $\int \sin^3 x \mathrm{d}x =$

Rules to Live By

- Always check your integration by differentiation
- Integration can be very tricky, but differentiation is easy!
- When integrating you can guess and waive your hands, but if you can verify your answer by differentiation it's the correct one!
- So once you have the answer, by hook or crook, check it by differentiation.

What Integration Constant?

- The integration constant is often determined by side conditions.
- Example:

$$f(x) = F'(x) = 4x^3, \qquad F(1) = 5.$$

More Examples