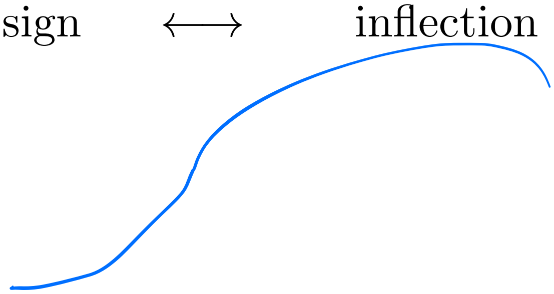
Math 1210-23

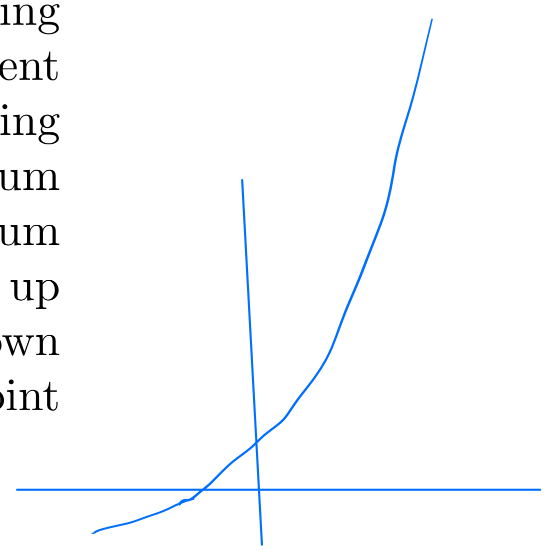
Notes of 3/12/24

3.5 Sophisticated Graphing

- **Unsophisticated Graphing:** compute a bunch of points on the graph and connect them somehow. That's what your calculator does.
- **Sophisticated Graphing:** Use symmetry, asymptotic behavior, first and second derivatives.
- Quick review of the relationship between derivatives and the shape of the graph of $y = f(x)$:

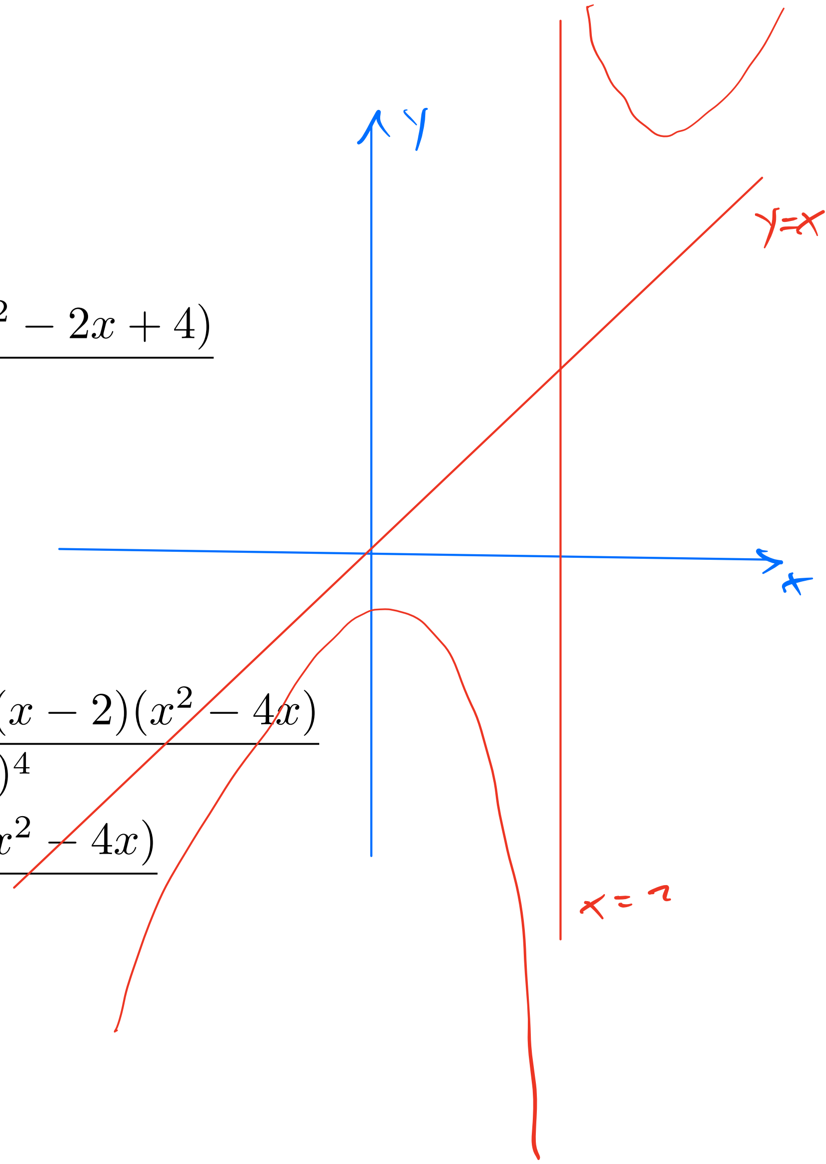
$f' > 0$	$\longleftrightarrow$	f is increasing
$f' = 0$	$\longleftrightarrow$	horizontal tangent
$f' < 0$	$\longleftrightarrow$	f is decreasing
$f' > 0 \longrightarrow f' < 0$	$\longleftrightarrow$	local maximum
$f' < 0 \longrightarrow f' > 0$	$\longleftrightarrow$	local minimum
$f'' > 0$	$\longleftrightarrow$	f is concave up
$f'' < 0$	$\longleftrightarrow$	f is concave down
f'' changes sign	$\longleftrightarrow$	inflection point





- We have already exploited these ideas and will again in the appropriate context.
- Let's discuss Example 2 from the textbook.

$$
\begin{aligned}
f(x) &= \frac{x^2 - 2x + 4}{x - 2} \\
&= x + \frac{4}{x - 2} \\
f'(x) &= \frac{(2x-2)(x-2) - (x^2 - 2x + 4)}{(x-2)^2} \\
&= \frac{x^2 - 4x}{(x-2)^2} \\
&= \frac{x(x-4)}{(x-2)^2} \\
f''(x) &= \frac{(2x-4)(x-2)^2 - 2(x-2)(x^2-4x)}{(x-2)^4} \\
&= \frac{(2x-4)(x-2) - 2(x^2-4x)}{(x-2)^3} \\
&= \frac{8}{(x-2)^3}
\end{aligned}
$$



$f(x) = \dfrac{1}{1+x^2}$

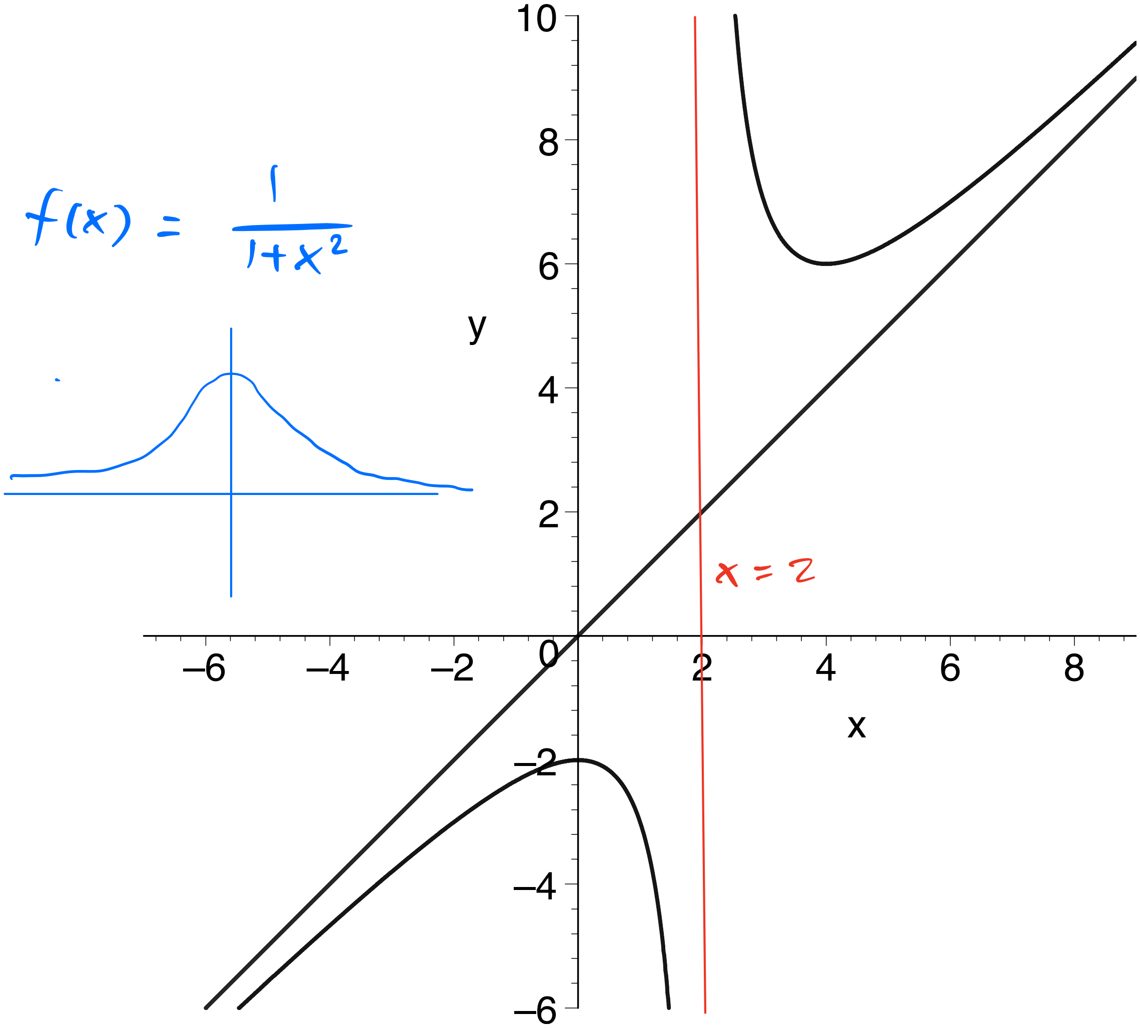
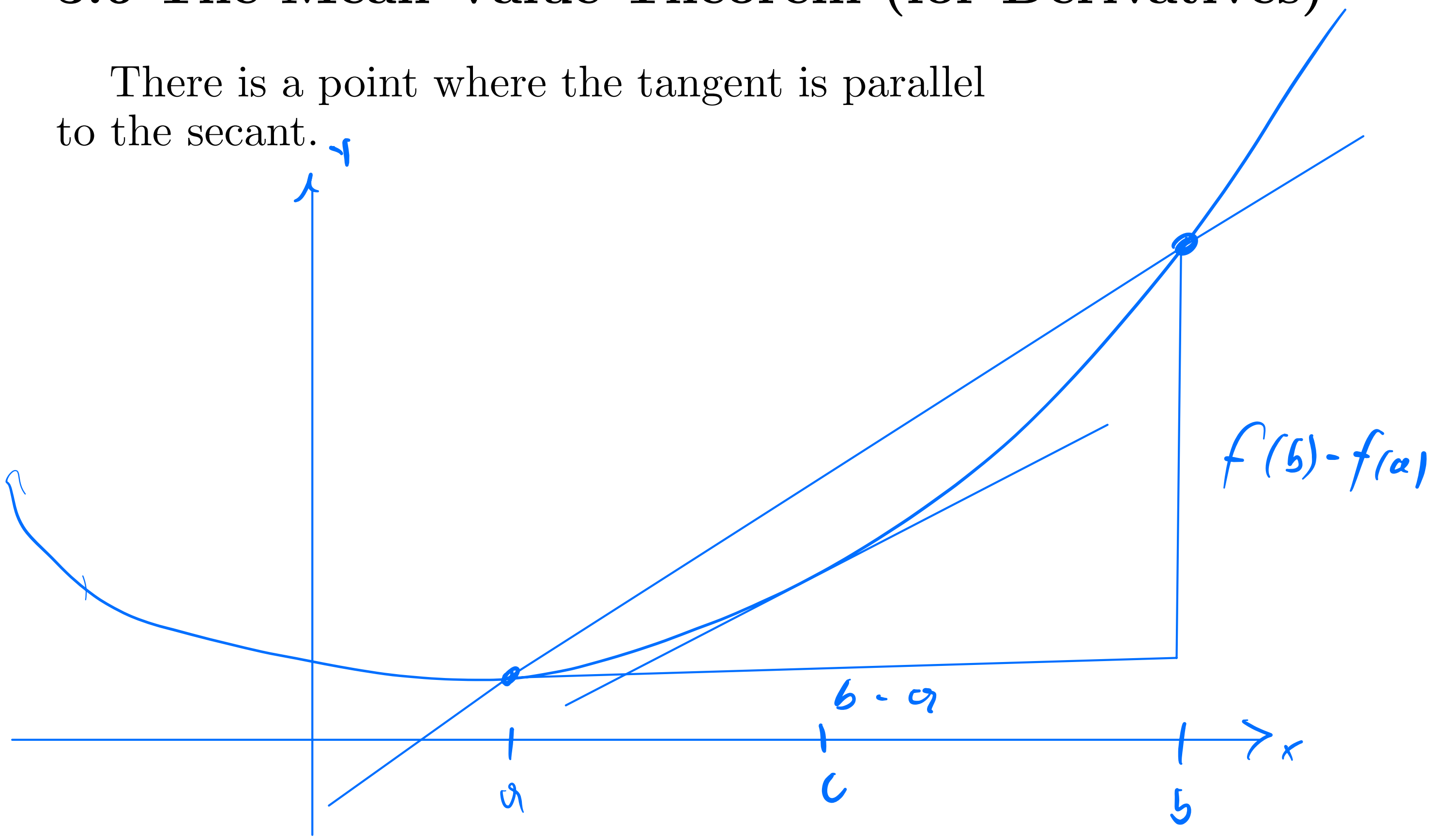


Figure 1. $f(x) = \frac{x^2-2x+4}{x-2}$.

MVT

3.6 The Mean Value Theorem (for Derivatives)

There is a point where the tangent is parallel to the secant.



Theorem A, page 186. If f is continuous on $[a, b]$ and differentiable on (a, b) then there is at least one number c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

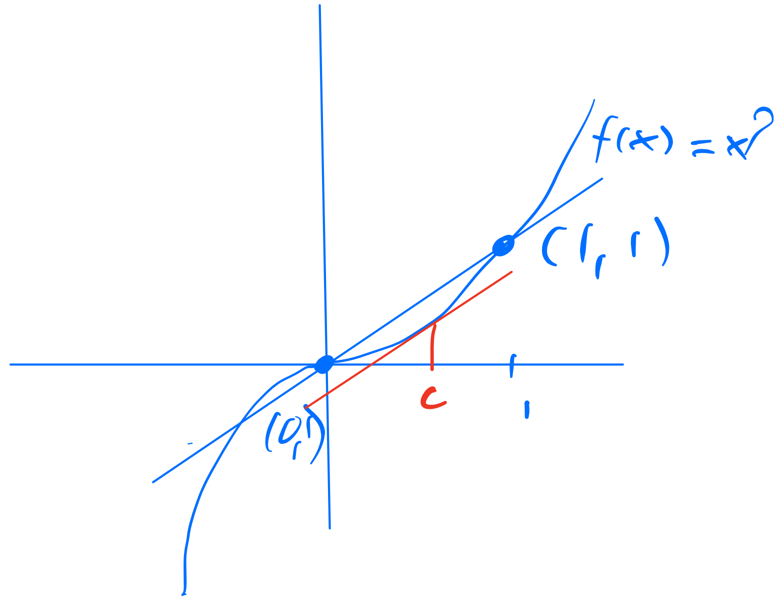
or, equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

- Example: $f(x) = x^3$, $[a, b] = [0, 1]$

$f'(x) = 3x^2$

$c = ?$



$$\frac{f(b) - f(a)}{b - a} = \frac{1 - 0}{1 - 0} = 1$$

$$= f'(c)$$
$$= 3c^2$$

$$3c^2 = 1$$
$$c^2 = \frac{1}{3}$$
$$c = \frac{1}{\sqrt{3}} > \frac{1}{2}$$

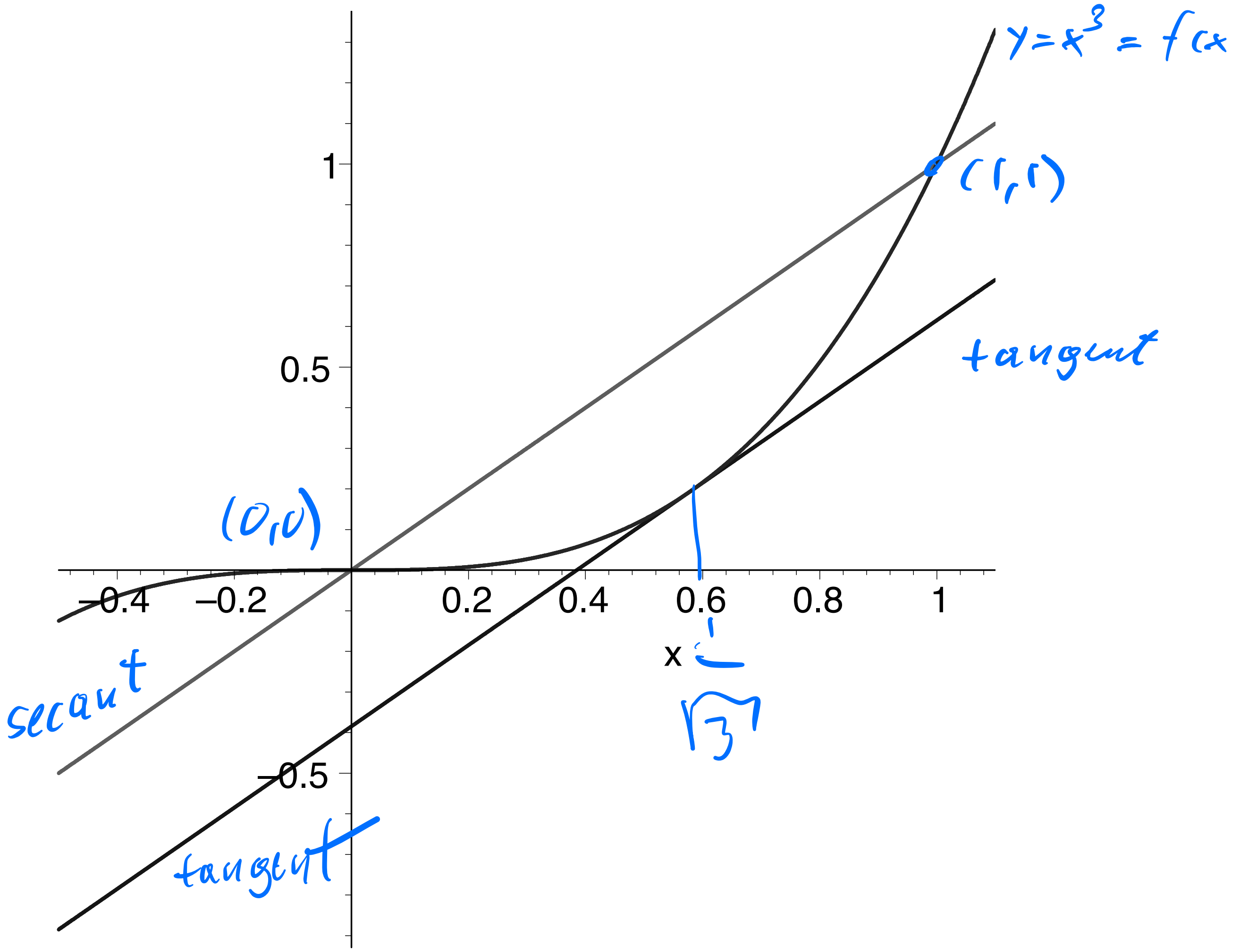
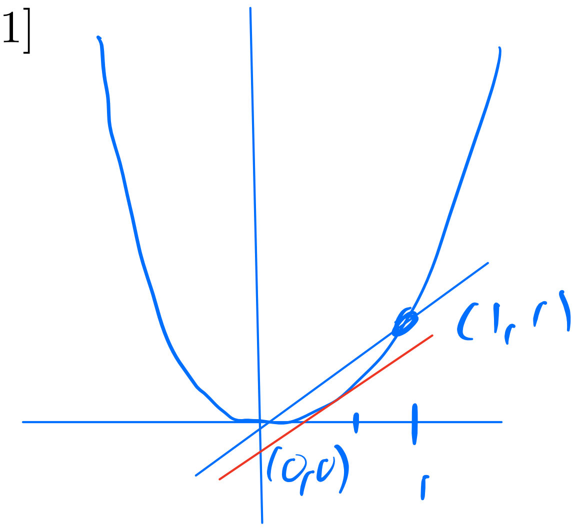


Figure 2. MVT, $f(x) = x^3$, $[a, b] = [0, 1]$, $c = \frac{1}{\sqrt{3}}$.

- Example: $f(x) = x^2$, $[a, b] = [0, 1]$

$f'(x) = 2x$

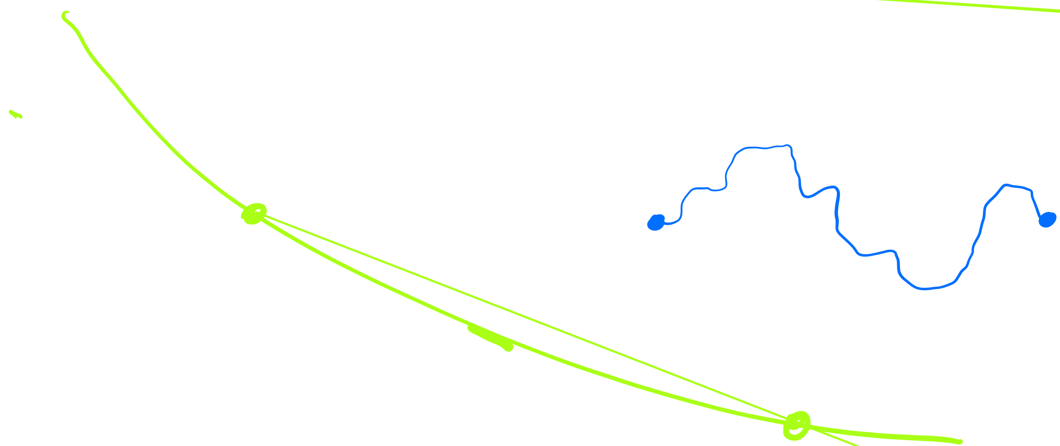


$$\frac{1-0}{1-0} = 1 = f'(c)$$

$$= 2c$$

$2c = 1$

$c = \frac{1}{2}$



$(a+b)(a-b) = a^2 - b^2$

$(b+a)(b-a) = b^2 - a^2$

- Example: f a quadratic polynomial, general interval.

$$f(x) = \alpha x^2 + \beta x + \gamma \qquad [a,b]$$

$$\frac{f(b)-f(a)}{b-a} = \frac{\alpha b^2 + \beta b + \gamma - (\alpha a^2 + \beta a + \gamma)}{b-a}$$

$$\frac{b^2-a^2}{b-a} =$$

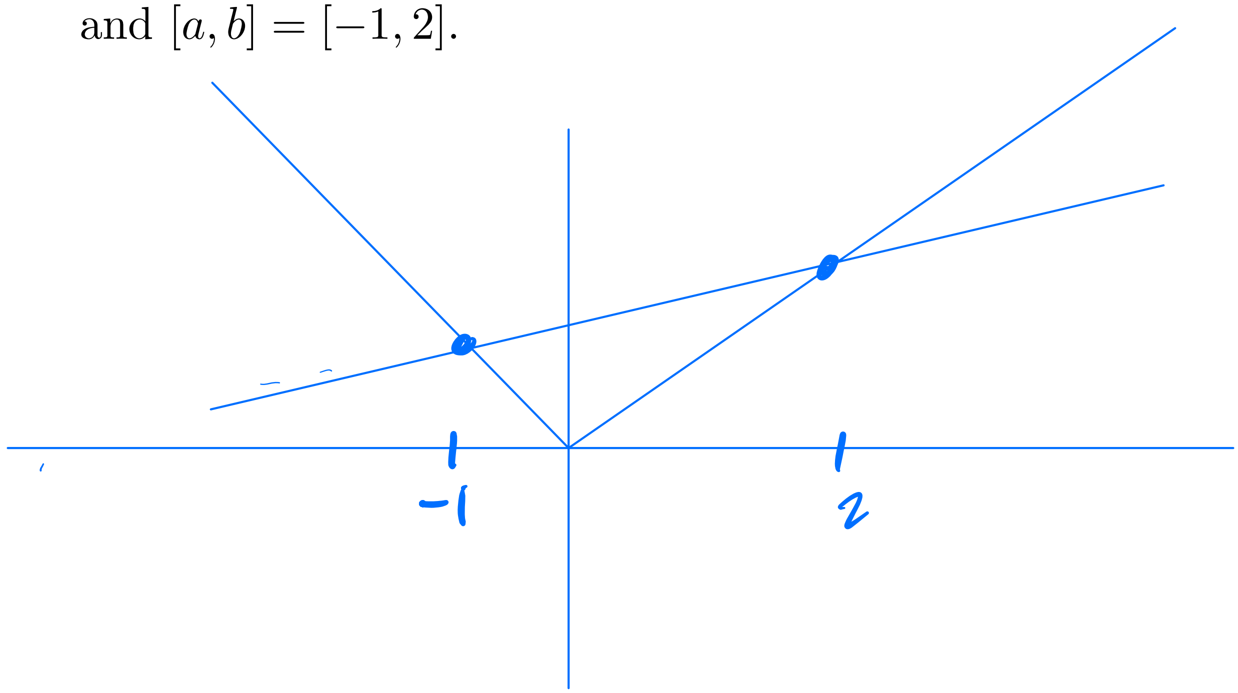
$$= \frac{\alpha(b^2-a^2) + \beta(b-a)}{(b-a))}$$

$$= \alpha\underbrace{(b+a)}_{2\frac{b+a}{2}} + \beta \qquad = f'(c) = 2\alpha c + \beta$$

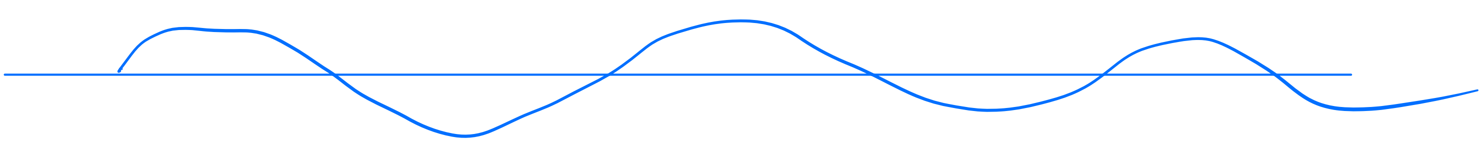
$$= \alpha\Big(\underbrace{2\frac{(b+a)}{2}}_{c}\Big) + \beta \qquad c = \frac{b+a}{2}$$

$$= 2\alpha c + \beta = f'(c)$$

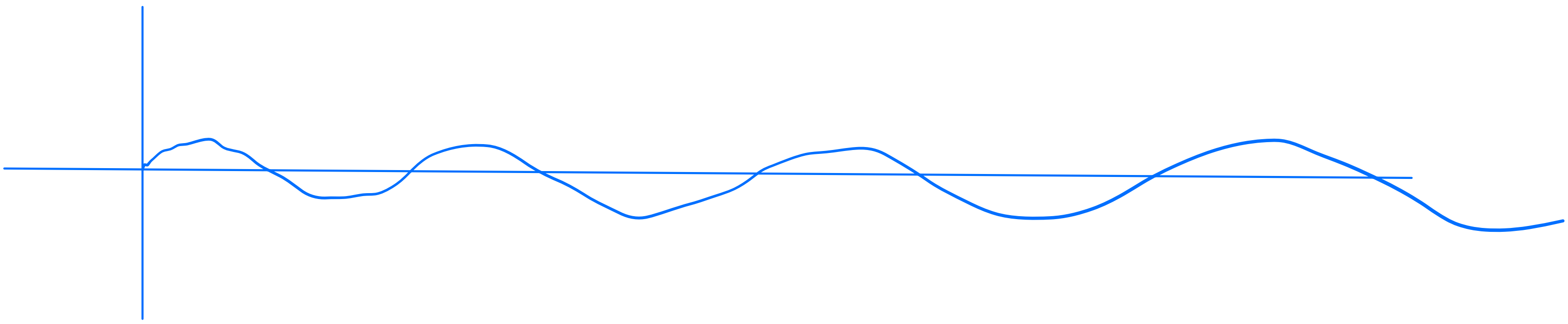
- If f is not differentiable at some point in (a, b) there may be no such c. Example: $f(x) = |x|$ and $[a, b] = [-1, 2]$.

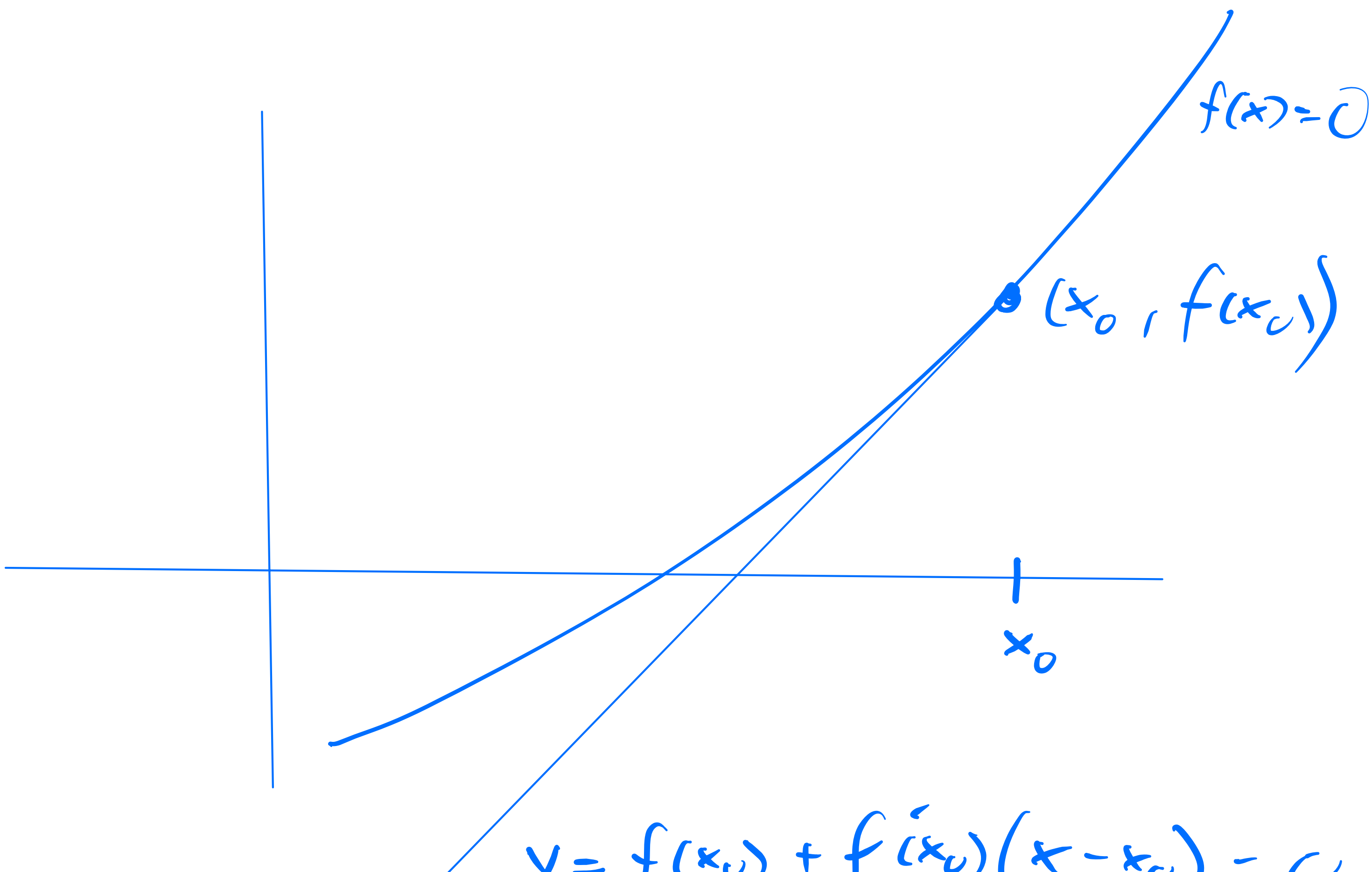


- There may be several several such values of c. Example: $f(x) = \sin x$ and $[a, b] = [0, 1000\pi]$



- It's hard to appreciate the significance of the MVT at this stage but we'll see a nifty application tomorrow.





$y = f(x_0) + f'(x_0)(x - x_0) = 0$

$y' = f(x_0)$

$$f'(x_0)(x - x_0) = -f(x_0)$$

$$x - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} = x_1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\ldots$$