Math 1210-4

Notes of 10/23/17

- Notes on Exam 2, each motivated by observing the same mistake more than once.

- **Problem 2:**
  \[(uv)' = u'v + uv' \neq u'v'\]

- **Problem 3:**
  \[
  \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \neq \frac{uv' - u'v}{v^2}
  \]
  You can remember the sign of the numerator of the derivative by observing that if the numerator of the function is increasing then the function is increasing, and so the \(u'\) term must have the positive sign. By the same token, if the denominator of the function is increasing then the function is decreasing, and so the \(v'\) term must have the negative sign. There were also lots of algebraic errors when converting to standard form!

- **Problem 3:** When you multiply with sums you need to write them in parentheses:

\[
\frac{d}{dx} \frac{x^4 + 2x}{x^2 - 1} = \frac{(4x^3 + 2)(x^2 - 1) - (x^4 + 2x)2x}{(x^2 - 1)^2}
\]

\[
\neq \frac{4x^3 + 2(x^2 - 1) - x^4 + 2x2x}{(x^2 - 1)^2}
\]
Problem 3: I saw the following calculation several times.

\[
\frac{d}{dx} \frac{x^4 + 2x}{x^2 - 1} = \frac{(4x^3 + 2)(x^2 - 1) - (x^4 + 2x)(2x)}{(x^2 - 1)(x^2 - 1)}
\]

\[
\neq \frac{(4x^3 + 2)(x^2 - 1) - (x^4 + 2x)(2x)}{(x^2 - 1)(x^2 - 1)}
\]

\[
= \frac{(4x^3 + 2) - (x^4 + 2x)(2x)}{(x^2 - 1)}
\]

and I have seen similar calculations before.

You must not do stuff like that!

- If you could you would have, for example

\[
-\frac{29}{33} = \frac{2 \times 3 - 5 \times 7}{3 \times 11} \neq \frac{2 \times \bullet - 5 \times 7}{\bullet \times 11} = \frac{2 - 5 \times 7}{11} = \frac{-33}{11} = -3.
\]

Fraction Rules apply to rational expressions as well!

- Problem 4:

\[(\sin t)^2 = \sin^2 t \neq \sin t^2 = \sin(t^2)\]

- Problem 4: This is somewhat of a cosmetic point, although it is also relevant for computer programming. Instead of

\[
\frac{d}{dt} \sin^2 t + \cos^2 t = 2 \sin t \cos t + -2t \sin t^2
\]
you should omit the plus sign:

\[
\frac{d}{dt} \sin^2 t + \cos^2 t = 2 \sin t \cos t - 2t \sin t^2.
\]

If you really want to indicate that you are adding something that has a negative factor you should write

\[
\frac{d}{dt} \sin^2 t + \cos^2 t = 2 \sin t \cos t + (-2t \sin t^2).
\]

- **Problem 6:** Once again:

\[
\frac{d}{dx} (\sin^2 \zeta + \cos^2 \zeta) = \frac{d}{dx} 1 = 0.
\]

- **Problem 10:** \(v\) is a velocity, not a distance. The speed at which the tip of the shadow is moving away from the lamp post equals the rate at which the shadow is growing, plus the speed at which you are walking!
3.4 More Extreme Value Problems

- You want to fence a triangular piece of land along a river with $L$ feet of fencing. You don’t need a fence along the river. How do you maximize the fenced in area? Assume your triangle is isosceles.

![Diagram of a triangle with sides $L/2$, height $h$, and base $x$]

\[ A = xh = \max_0^{L/2} \quad \frac{L^2}{4} \]

\[ x^2 + h^2 = \frac{L^2}{4} \]

\[ h^2 = \frac{L^2}{4} - x^2 \]

\[ h = \left(\frac{L^2}{4} - x^2\right)^{1/2} \]

\[ A = x \left(\frac{L^2}{4} - x^2\right)^{1/2} \]
\[ A' = \left( \frac{L^2}{4} - x^2 \right)^{\frac{3}{2}} + x \frac{L^2}{2} \left( \frac{L^2}{4} - x^2 \right)^{-\frac{3}{2}} \left( -2x \right) = 0 \]

\[ \left( \frac{L^2}{4} - x^2 \right)^{\frac{3}{2}} - x \left( \frac{L^2}{4} - x^2 \right)^{-\frac{3}{2}} = 0 \]

\[ \pm \left( \frac{L^2}{4} \frac{2}{\kappa} - \frac{\kappa}{4} \right)^{\frac{3}{2}} \]

\[ \frac{L^2}{4} - x^2 - x^2 = 0 \]

\[ 2x^2 = \frac{L^2}{4} \]

\[ x^2 = \frac{L^2}{8} \]

\[ \sqrt[3]{h} = \frac{2\sqrt{2}}{L} \]

\[ x = \frac{L}{\sqrt[3]{h}} = \frac{\sqrt[3]{2} L}{8} = \frac{\sqrt[3]{2} L}{4} \]

\[ A = \max \]

\[ A = xh \]

\[ A^2 = \int (x) = x^2 h^2 = x^2 \left( \frac{L^2}{4} - x^2 \right) = \max \]

\[ = \frac{L^2}{2} x^2 - x^4 \]

\[ f'(x) = Lx - 4x^3 = 0 \]

\[ = x (2L - 4x^2) = 0 \]

\[ x^2 = \frac{L^2}{8} \]
Example 4: A fish swims upstream with velocity $v$ relative to the water. The current of the river is $-v_c$. The energy expended in traveling a distance $d$ up the river is directly proportional to the time required to travel the distance and the cube of the velocity. What velocity minimizes the energy?

Query: What about swimming downstream?

If you don’t care about fish swimming against the current, think of rowing a boat going upstream.

$$E = k \frac{v^3 t}{v - v_c}$$

$$t = \frac{d}{v - v_c}$$

$$= k \frac{v^3 d}{v - v_c} = kd \frac{v^3}{v - v_c} = \min$$

(WLOG) $kd = 1$

$$\Rightarrow \frac{v^3}{v - v_c} = f(v) = \min$$

$$f(v) = kd \frac{v^3}{v - v_c}$$

$$f'(v) = kd \frac{3v^2(v - v_c) - v^3}{(v - v_c)^2} = 0$$
$$3v^2(v - v_c) - v^3 = 0 \quad \text{distribute}$$

$$3v^3 - 3v^2v_c - v^3 = 0 \quad \text{divide by } 3v^2v_c$$

$$2v^3 = 3v^2v_c \quad \div v^2$$

$$2v = 3v_c$$

$$v = \frac{3v_c}{2}$$
• Example 5. A 6-foot hallway makes a right turn. What is the length of the longest thin rod that can be carried around the corner, assuming you cannot tilt the rod. (Think of a sheer of plywood or glass).

• Expectations

\[ \theta = 45^\circ \]

\[ L = 2 \sqrt{72} \]

\[ L = x + y = \min \]

\[ \frac{u}{x} = \sin \theta \]

\[ \frac{v}{y} = \cos \theta \]
\[
x = \frac{u}{\sin \theta} \quad \gamma = \frac{v}{\cos \theta}
\]

- OK, that was easy. But now supposes that the hall changes width, say from \(u\) feet to \(v\) feet.

\[
f(\theta) = x + \gamma
\]

\[
= \frac{u}{\sin \theta} + \frac{v}{\cos \theta}
\]

\[
f'(\theta) = \frac{-ucos\theta}{\sin^2 \theta} + \frac{v \sin \theta}{\cos^2 \theta} = 0
\]

\[
\frac{ucos\theta}{\sin^2 \theta} = \frac{v \sin \theta}{\cos^2 \theta}
\]

\[
u \cos \beta = v \sin \beta \quad \frac{\cos^3 \theta}{\sin \beta} = v
\]

\[
u = \frac{\sin^3 \theta}{\cos^3 \theta} = \tan^3 \theta \quad \left( \frac{\theta}{3} \right)
\]

\[
\tan \theta = \left( \frac{u}{v} \right)^{1/3}
\]

\[
\theta = \arctan \left( \frac{u}{v} \right)^{1/3}
\]
• If time permits: You want to build a square box holding a volume $V$. The material for the top is 10 times as expensive as the material for the bottom and the four sides. Let $h$ be the height of the top and $s$ the length of a side. What is the ratio $h/x$ that minimizes the cost?