Math 1210-23, Spring 2024

Notes of 3/11/24

Notes on Exam 2

General Notes

- Several people tried to ask questions or talk with me during the exam. I can't talk with you because it disrupts and distracts people. Moreover, if we have a quick whispered conversation there is an excellent chance that I don't even understand what you are asking, or I misunderstand and give you a misleading answer. Also, you may think you are asking a quick yes or no question, but I may not agree. In the unlikely and rare event that you have an exam that's blank or illegible just walk up to the front desk and show it to me. If you think a question is inconsistent write a note and you'll get generous credit. You can also email me after the exam and I'll respond quickly.

Most people left the exam early. In my experience there is no clear correlation between the score on an exam and the time that a student spends on it. But I recommend strongly that you go slowly and carefully, and you check your answers. Careless mistakes happen. That's OK, but it's not OK not to catch them. I saw many algebraic and arithmetic errors. They could have been caught or avoided by working slowly and deliberately, and by checking and rechecking your answer.

With all due respect, I believe that you are not spending your time effectively if you rush through the exam, and leave the classroom as soon as you possibly can.

- It does not make sense to be late for an exam!
- Please enter only the answers, and no calculations, in the boxes provided on the exam!

- This was written as I was grading, and my comments are mostly about the mistakes that I saw, but of course keep in mind that most answers I saw were actually correct!
- This is like driver's ed: most of the time is spent on discussing how to avoid accidents, despite accidents being very rare in routine driving. They still must be avoided, however!
- For your information, here is a Table of the average point counts for each question:

Question:	1	2	3	4	5	6	7	8	9
Average Score:	9.4	9.3	7.6	5.4	8.4	7.3	4.9	5.8	7.0

- The mean score of the exam was 66 points (73%). That's good, but it could be better! 8 people received 100% (90 points) on the exam (great job!), and 18 people received an A (81 points or more).
- The first four questions are straight differentiation questions (of increasing complexity).
- The last question consisted of 10 T/F statements. The mean score was 7.
- The current grade distribution in this class is:

$\geq 90\%$	$\geq 85\%$	$\geq 80\%$	$\geq 75\%$	$\geq 70\%$	$\geq 65\%$	$\geq 60\%$
A	$A-$	$B+$	B	$B-$	$C+$	C
24	30	10	19	17	8	10

$\geq 55\%$	$\geq 50\%$	$\geq 45\%$	$\geq 40\%$	$\geq 0\%$
$C-$	$D+$	D	$D-$	E
13	5	2	3	8

Exam Answers

Following is a rendering of the exam answers, with additional comments indicated by the Danger symbol.

Math 1210-23 Spring 2024 Exam 2 Answers

As usually, this answer sheet gives you more information than you needed to provide on the exam.

-1- (hw 4, problem 8, mean score = 9.4.) If

$$f(x) = 2x^2 - 8x + 4$$

then

$$f'(x) = 4x - 8$$

and

$$f'(2) = 0$$

I was amazed by the number of people (maybe 20) who evaluated $f'(2)$ as 8 or 4. That is the kind of error that can be much reduced by going slowly, and that can be caught by checking your answers!

There were a few answers that had an x in the second box. That can't possibly be true, if you evaluate a function at a specific number the result cannot possibly contain the variable! You want to be alert to patterns like this!

-2- (Notes of 1/26/24, mean score = 9.3.) Given

$$f(x) = (x + 2)(x + 3)$$

compute $f'(x)$

Answer: This can be computed either by the product rule, or by first expanding the polynomial. Either way, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}(x+2)(x+3) = 1 \times (x+3) + (x+2) \times 1 = 2x+5$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}(x+2)(x+3) = \frac{\mathrm{d}}{\mathrm{d}x}(x^2+5x+6) = 2x+5$$

A number of people simply expanded the given expression (sometimes incorrectly) and did not differentiate at all.

-3- (hw 4, problem 9, mean score = 7.6.) If

$$f(x) = \frac{x^2}{1+x^2}$$

then

$$f'(x) = \frac{2x}{(1+x^2)^2}.$$

Answer: We get, by the quotient rule,

$$f'(x) = \frac{2x(1+x^2) - x^2 \times 2x}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}.$$

I saw numerous algebra errors. If you are likely to make algebraic mistakes, go more slowly, check your answers, and practice!

In particular note that

$$(1+x^2)^2 \neq 1+x^4.$$

I saw a few answers that were constants, like 0 or 1. This can't possibly be true! If that answer was actually

$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$

correct then the function you differentiated would have to be a polynomial.

A number of people expanded the denominator:

$$(1+x^2)^2 = 1 + 2x^2 + x^4.$$

That is correct, and I did not deduct points, but it's impractical. If you have a polynomial that is already in factored form you want to unfactor it only if you have a good reason (like you are adding two polynomials). The reason for this is that unfactoring is easy and mechanical, but factoring can be tricky!

-4- (Notes of 1/30/24, mean score = 5.4.) Compute

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin^2 x - \sin x^2.$$

Answer: The main issue in this problem is to understand that in the first term we first compute the sine and then square, and in the second we first square and then evaluate the sine. Using the sine rule, the product rule, linearity of differentiation, and the chain rule, we get

$g'(x)$ $f'(g(x))$

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin^2 x - \sin x^2 = 2 \sin x \cos x - 2x \cos x^2.$$

$(\cos x^2)(2x)$

A lot of people had trouble with this question. The chain rule was frequently misapplied, and there were also some trig errors.

-5- (Notes of 2/12/24, mean score = 8.4.) Recall that the Unit Circle is defined by the equation

$$x^2 + y^2 = 1. \qquad (1)$$

Think of y as a function of x, differentiate implicitly, solve for y' and express y' in terms of x and y.

Answer: Differentiating implicitly in (1) gives

$$2x + 2yy' = 0$$

and solving for y' gives the answer:

$$y' = -\frac{x}{y}.$$

This was also difficult for many people.

-6- (Notes of 2/5, mean score = 7.3.) Suppose you want to solve

$$f(x) = x^2 - 2 = 0$$

by Newton's Method, starting with $x_0 = 1$. Compute x_1.

Answer: Newton's Method is defined by

$$x_0 \text{ given}, \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots. \quad (2)$$

$\cancel{x_{n+1} = -\frac{f(x_n)}{f'(x_n)}}$

In this case,

$$f(x) = x^2 - 2 \quad \text{and} \quad f'(x) = 2x.$$

Substituting $x_0 = 1$ in (2) give

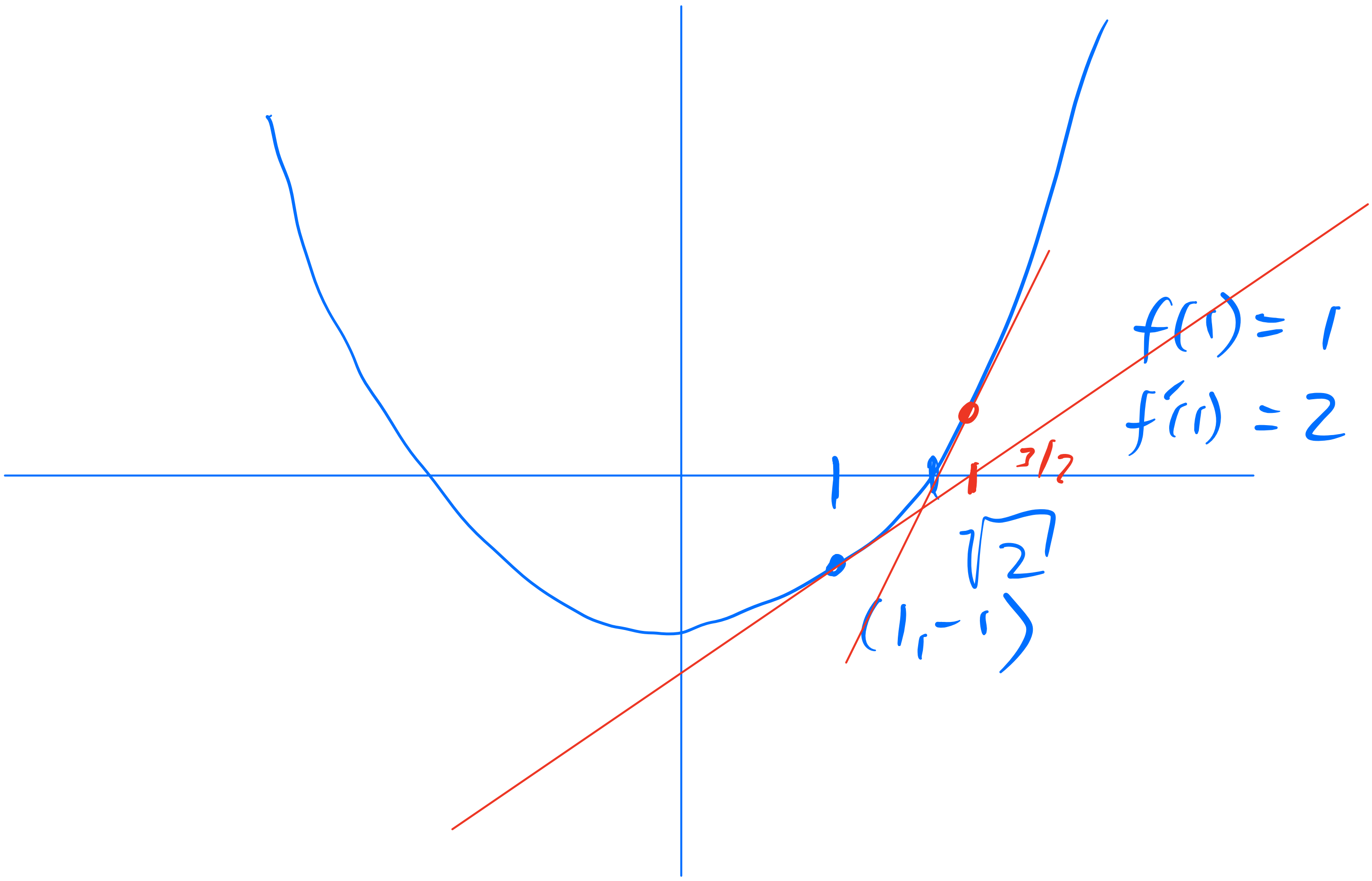
$$x_1 = 1 - \frac{1^2 - 2}{2} = \frac{3}{2}.$$

A number of people had the wrong formula for Newton's method. You are in good company. Long ago I took an exam with basically the same question. I misremembered the formula, got the wrong answer, and shudder to this day when I think back to it.

The basic idea of Newton's Method is to find the x-intercept of the tangent at the current point. If you did

$$f(x) = x^2 - 2 = 0 \qquad x_0 = 1$$

$$f'(x) = 2x$$



$$\frac{y-(-1)}{x-1} = 2$$

$$y+1 = 2(x-1) = 2x-2$$

$$y = 2x-3 = 0$$

$$x = \frac{3}{2}$$

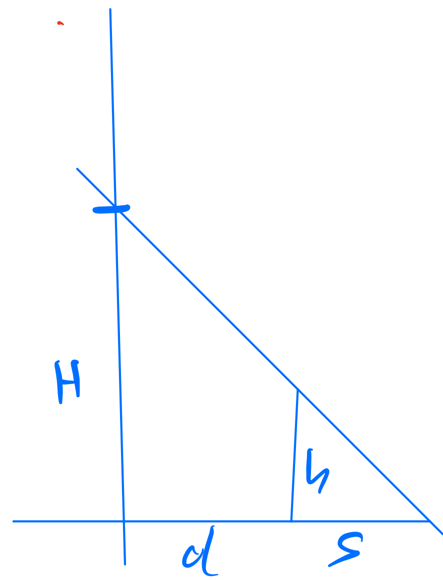
not remember the formula for Newton's method you could have used this idea to derive it.

In this case, we start at $x_0 = 1$, and we have $f(1) = -1$ and $f'(1) = 2$. So we compute the equation of the line that passes through the point $(1, -1)$ and has the slope 2, and then find the x-intercept of that line.

I recommend that you use fractions instead of decimals, i.e., $\frac{3}{2}$ instead of 1.5, and avoid mixed numbers like $1\frac{1}{2}$ altogether.

I also saw many arithmetic errors.

-7- (Notes of 2/14/24, mean score = 4.9.) You are h feet tall and you are walking away from a street light that is H feet tall. Assume that $H > h$. Suppose you are walking at a steady rate of v feet per minute. Compute the speed V at which the tip of the shadow is moving away from the street light.



Proceeding as we did in class (see the notes of February 14), letting s equal the length of the shadow, and d your distance from the lamp post, and using similar triangles we get

$$\frac{s}{h} = \frac{s+d}{H}. \tag{3}$$

Differentiating gives

$$\frac{s'}{h} = \frac{s'+d'}{H} = \frac{s'+v}{H}. \tag{4}$$

Solving for s' gives

$$s' = \frac{h}{H-h} v \tag{5}$$

The shadow is growing at the rate s'. However, we want to know how fast its tip is moving away from the lamp post, so we need to add v. So the speed V at which the tip of the shadow is moving is

$$V = v + s' = v + \frac{h}{H-h}v = \left(1 + \frac{h}{H-h}\right)v = \frac{H}{H-h}v. \tag{6}$$

This is a typical problem where it is useful to be mindful of the dimensions involved. For example, an expression like $\frac{1}{h}+v$ cannot possibly occur in your calculation since you can't add the reciprocal of a length to a velocity. The answer is a velocity.

-8- (Notes of 2/16/24, mean score = 5.8.) Suppose you manufacture cubes of length (and width and height) s. Use differentials to show that if your equipment causes an error of p percent in the length s then the error in the volume of the cube will be approximately x percent. What is x?

Answer: The volume of a cube with side length s is $V = s^3$. In class we discussed that if the error in s is Δs then the corresponding error ΔV in V is given by

$$\Delta V = \frac{\mathrm{d}V}{\mathrm{d}s}\Delta \cancel{s} = 3s^2 \Delta s.$$

s

Dividing by $V = s^3$ gives

$$\frac{\Delta V}{V} \approx \frac{3s^2}{s^3} = 3\frac{\Delta s}{s}.$$

Thus an error of p percent in s will cause an error of approximately $3p$ percent in V. The relative error in the volume equals approximately 3 times the relative error in the side length.

The answer cannot possibly contain x.

The answer depends on p. If p does not occur in your answer it can't be right!

-9- (hw 7, mean score = 7.0.) The last problem has 10 simple true/false questions. Mark each statement by circling **T** (true) or **F** (false) as appropriate. If the item mentions a function then x denotes the independent variable of that function. You do not need to give reasons for your answers.

Numbers in parentheses refer to the question number on pages 147-148 of the textbook.

False The tangent line to a curve can touch the curve at only one point. False, consider, for example, $f(x) = 1+\cos x$ and the x axis. (2)

True It is possible for the speed of an object to be increasing while the velocity is decreasing. True, speed is the absolute value of velocity, and if the velocity is negative, then decreasing velocity means that the velocity is getting more negative and the speed is therefor increasing. (6)

False If $f'(x) = g'(x)$ for all x then $f(x) = g(x)$ for all x. False, f and g may differ by a non-zero constant. (8)

True If $y = (x^3 + x)^8$ then $D_x^{25} y = 0$. True, $(x^3 + x)^8$ is a polynomial of degree 24 and its 25-th derivatives is zero. (18)

True The derivative of a rational function is a rational function. True, you can see this examining the quotient rule, noting that a rational function is the ratio of two polynomials, and observing that the derivative of a polynomial is a polynomial. (20)

True If $h(x) = f\big(g(x)\big)$ where f and g are both differentiable, and $g'(c) = 0$, then $h'(c) = 0$. True, this follows from the chain rule, since $g'(c)$ is a factor of $h'(c)$. (24)

True If the radius of a circle is increasing at 4 feet per second, then the circumference of the circle is increasing at 8π feet per second. True, the circumference $c = 2\pi r$, $r' = 4$, and $c' = 2\pi r'$. (28)

True If $s = 5t^3 + 6t - 300$ gives the position of an object on a horizontal coordinate line at time t, then the object is always moving to the right. True, the derivative is always positive. (32)

True If water is being pumped into a spherical tank at a rate of 3 gallons per second, then the height of the water in the tank will increase more and more rapidly as the tank nears being full. True, this follows from the geometry. (34)

False If $y = x^5$ then $\mathrm{d}y \geq 0$. False, $\mathrm{d}y = 4x^4\mathrm{d}x$, and $\mathrm{d}x$ might be negative. (36)

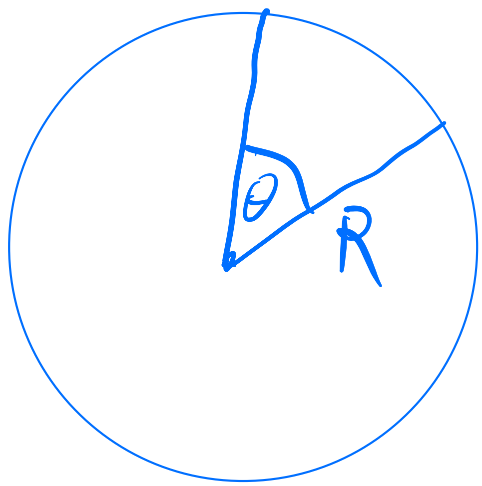
I was impressed by the large number (21 out of 134) of completely correct answers!

Review

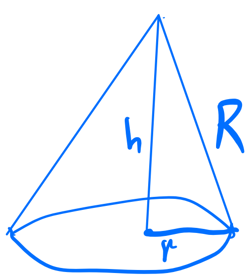
- Once again, here is a reminder of our differentiation rules:

$$
\boxed{
\begin{array}{rcll}
x^r & = & rx^{r-1} & \textbf{Power Rule} \\
(f+g)' & = & f'+g' & \textbf{Sum Rule} \\
(f-g)' & = & f'-g' & \textbf{Difference Rule} \\
(kf)' & = & kf' & \textbf{Constant Multiple Rule} \\
\sin x & = & \cos x & \textbf{Sine Rule} \\
\cos x & = & -\sin x & \textbf{Cosine Rule} \\
(uv)' & = & u'v+uv' & \textbf{Product Rule} \\
\left(\frac{u}{v}\right)' & = & \frac{u'v-uv'}{v^2} & \textbf{Quotient Rule} \\
f\big(g(x)\big) & = & f'\big(g(x)\big)g'(x) & \textbf{Chain Rule}
\end{array}
}
\tag{7}
$$

If time allows, let's do another Optimization Problem. You form a cone by cutting a sector with angle θ out of a circle with radius R and bending what's left into a cone. How do you choose θ to maximize the volume of the cone? This can be done in a very complicated way or in a much simpler way.



$\theta = ?$



$$V = \frac{1}{3}\pi r^2 h = max$$

$$2\pi r = R(2\pi - \theta)$$

$$r^2 + h^2 = R^2$$

$$R^2 - h^2 = r^2$$

$$V = \frac{1}{3}\pi\left(R^2 - h^2\right)\underline{h}$$

$$\frac{d}{dh}V = \frac{1}{3}\pi\left(-2h^2 + \left(R^2 - h^2\right)\right)$$

$$= \frac{1}{3}\pi\left(R^2 - 3h^2\right) = 0$$

$$R^2 = 3h^2$$

$$h^2 = \frac{R^2}{3}$$

$$h = \frac{R}{\sqrt{3}}$$