Math 1210-23

Notes of 2/28/24

Chapter 2 Summary

- Exam 2 in our class will take place on Friday, March 1, 2024. It will have 9 questions and cover chapter 2 of our textbook. Five of the questions are straight differentiation problems, one is on Newton's Method, one is a related rates problem, one is on differentials, and one is a bunch of true/false questions.

The Subject

- As usual, the following list is neither complete nor self contained. Rather it is meant to trigger your memory and activate your comprehension. If any of these points are not clear to you make sure you review the relevant material before the exam. You want to understand everything that's indicated here but not everything will be covered by the exam.

You want to have a thorough grasp of the **concept of a derivative**:

- It's the slope of the tangent.
- It's the limit of the slopes of the secants.

- It measures how rapidly a function is changing.
- The derivative of location is velocity
- The derivative of velocity is acceleration
- The derivative is defined as

$$
\begin{aligned}
f'(x) &= \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{\Delta x \longrightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{z \longrightarrow x} \frac{f(z) - f(x)}{z - x}
\end{aligned}
\tag{1}
$$

- There are a large number of **notations**, including the following. Suppose that $y = f(x)$.

$$
f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = Dy = Df(x) = D_x f(x) = D_x y. \tag{2}
$$

  You should be familiar, and in fact comfortable, with all of those notations, and be able to use whichever is most appropriate or convenient for a given problem.

  Whenever you differentiate you need to be clear in your mind about the variable with respect to which you are differentiating, and you need to be aware what other variables depend on that variable, and which are constant. For example, if y is a function of x and D or prime denotes differentiation with respect to x, then

$$Dx^2 = 2x \quad \text{and} \quad Dy^2 = 2yy'.$$

- To compute derivatives we apply their properties, i.e., the **differentiation rules** shown in the box on the next page. You don't want to memorize these formulas! Instead you want to use them so often that you can't possibly forget them, and you want to be able to derive them and explain them to your friends.

$$
\begin{array}{rcll}
\frac{\mathrm{d}}{\mathrm{d}x} x^r & = & r x^{r-1} & \textbf{Power Rule} \\
(f+g)' & = & f' + g' & \textbf{Sum Rule} \\
(f-g)' & = & f' - g' & \textbf{Difference Rule} \\
(kf)' & = & kf' & \textbf{Constant Multiple Rule} \\
\frac{\mathrm{d}}{\mathrm{d}x} \sin x & = & \cos x & \textbf{Sine Rule} \\
\frac{\mathrm{d}}{\mathrm{d}x} \cos x & = & -\sin x & \textbf{Cosine Rule} \\
(uv)' & = & u'v + uv' & \textbf{Product Rule} \\
\left(\frac{u}{v}\right)' & = & \frac{u'v - uv'}{v^2} & \textbf{Quotient Rule} \\
\frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) & = & f'\big(g(x)\big)g'(x) & \textbf{Chain Rule}
\end{array} \tag{3}
$$

- The sum and constant multiple rules together mean that **differentiation is linear**.

- You should be able to apply these rules in combination, and differentiate anything that can in fact be differentiated by these rules.
- Apply the **onion method**: Ask what is the nature of the outermost layer? Specifically, what is the last operation when *evaluating* a given expression? Then apply the appropriate rule, and proceed similarly to differentiate subexpressions.
- Example:

$$f(x) = \frac{x\sin^2 x^2}{\sqrt{1+x^2}} \quad = \boxed{\frac{\boxed{\boxed{x}\,\boxed{\left(\boxed{\sin\boxed{x^2}}\right)^2}}}{\boxed{\sqrt{\boxed{\boxed{1}+\boxed{x^2}}}}}} \tag{4}$$

- Differentiation can be repeated, giving rise to higher derivatives, for example:

$$
\begin{aligned}
f(x) &= 2x^5 + 3x^4 - 4x^3 + 5x^2 - 6x + 7 \\
f'(x) &= 10x^4 + 12x^3 - 12x^2 + 10x - 6 \\
f''(x) &= 40x^3 + 36x^2 - 24x + 10 \\
f'''(x) &= 120x^2 + 72x - 24 \\
f^{(4)}(x) &= 240x + 72 \\
f^{(5)}(x) &= 240
\end{aligned} \tag{5}
$$

- Note how each differentiation reduces the degree of the polynomial by 1.

- In general, a function f is a polynomial of degree up to n if and only if the $(n+1)$-th derivative of f is everywhere zero.

- Differentiation can be done *implicitly*. For example, thinking of y as a function of x, we get

$$x^2+y^2 = 1 \quad \Longrightarrow \quad 2x+2yy' = 0 \quad \Longrightarrow \quad y' = -\frac{x}{y} \tag{6}$$

  This process is called **implicit differentiation**.

- Implicit differentiation occurs frequently in **Related Rates Problems**: Understand the problem, introduce variables, write one or more equations that hold at all time, differentiate, obtain equations that involve rates (derivatives), solve for what you want to know, make sure the dimensions in your results are consistent, substitute numbers only at the end if at all.

- **Differentials**: The change in a function value is approximately equal to the change in the independent variable, multiplied with the derivative. This is expressed as

$$\Delta y \approx \mathrm{d}y = f'(x)\mathrm{d}x = f'(x)\Delta x \qquad (7)$$

where in this context $\mathrm{d}x$ and $\mathrm{d}y$ are variables called *differentials* and Δx and Δy are the corresponding changes in x and y. Note that $\Delta x = \mathrm{d}x$ but in general $\Delta y \neq \mathrm{d}y$.

- For example, suppose you are producing cubes with a side length s. The volume of the cube is $f(s) = s^3$. A change Δs in the cube produces a change of

$$\Delta V \approx f'(s)\mathrm{d}s = 3s^2\mathrm{d}s = 3s^2\Delta s \qquad (8)$$

in the volume.

- Sometimes we are interested in the relative error, which is the percentage, except for a factor 100. Dividing by V in (8) gives

$$\frac{\Delta V}{V} \approx \frac{3s^2\mathrm{d}s}{s^3} = 3\frac{\mathrm{d}s}{s}.$$

In other words, an error of p percent in the side length causes an error of approximately $3p$ percent in the volume.

- **Linear Approximation.** The linear approximation of a function f at a point $\big(x_0, f(x_0)\big)$, say, is simply the tangent. It is given by

$$T(x) = f(x_0) + f'(x_0)(x - x_0) \tag{9}$$

- For example, if

$$f(x) = \sqrt{x}$$

and $x_0 = 400$ then

$$f'(x) = \frac{1}{2\sqrt{x}}$$

and, for example,

$$\begin{aligned}
\sqrt{(}401) &\approx f(400) + f'(400)(401 - 400 \\
&= \sqrt{400} + \frac{1}{2\sqrt{400}} \times 1 \\
&= 20 + \frac{1}{40} \\
&= 20.025
\end{aligned}$$

- More accurately, the square root of 401 is about 20.024984, so 20.025 is a pretty good approximation that you can compute in your head.

- In a profound sense (which, however, is beyond our scope) the approximation by the tangent is the better the closer x is to x_0.

- Don't confuse a differentiation rule such as the product rule
$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} \tag{10}$$
with the corresponding differential statement
$$\mathrm{d}(uv) = u\mathrm{d}v + v\mathrm{d}u. \tag{11}$$

Major error sources in differentiation:

- Failure to appreciate with respect to which variable (the *independent variable*) you differentiate, and which other variables depend on the independent variable.
- Misapplication of the product or quotient rules: The derivative of a product (or quotient) does not equal the product (or quotient) of the derivatives.
- Misapplication of the chain rule. You need to be clear on the sequence in which you apply functions. For example:

$$\begin{aligned} \sin^2 x &= (\sin x)^2 & \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sin^2 x &= 2\sin x\cos x \\ \sin x^2 &= \sin(x^2) & \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sin x^2 &= 2x\cos x^2 \end{aligned} \tag{12}$$

- Algebraic errors. These occur in particular when simplifying an expression, for example in preparation for computing a higher order derivative.

Newton's Method

- Newton's Method is a powerful technique to approximate a solution of

$$f(x) = 0 \tag{13}$$

  where f is a differentiable function.

- You start with an approximation x_0. The choice of x_0 depends on the problem, There is no universal choice (like $x_0 = 0$) that works in all cases!

- The basic idea of Newton's Method is to compute the tangent at the current point, and make the new point the point where the tangent (rather than the function) intersects the x-axis. This is best explained with a picture:

- The equation of the tangent at the point $\big(x_0, f(x_0)\big)$ is given by

$$T(x) = f(x_0) + f'(x_0)(x - x_0)$$

- Setting $T(x) = 0$ and solving for x gives

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- That value becomes the new approximation of the solution of (13):

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

- In general, Newton's method is defined by:

$$x_0 \text{ given}, \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

- The iteration is terminated when the approximation is sufficiently accurate. Often, this means that x_n does not change any more within the accuracy that is being used.

Notation and Writing

- Unlike in WeBWorK which will accept a correct answer in any form, write polynomials in factored form, or in standard form. The standard form of a polynomial of degree n is, of course:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0. \quad (14)$$

- Here are some suggestions for writing your work and answers:
- Use fractions instead of decimals.
- Avoid mixed numbers
- Cancel common factors in numerator and denominator.
- Unfactor a polynomial only if you have a good reason to do so. (For example, when you want to combine it with another polynomial.)
- It's OK to have a radical in the denominator.
- Use equals signs, for example when solving an equation, or when simplifying an expression.
- **Most important: write clearly, and explain what you are doing!**