Math 1210-23

Notes of 2/27/24

- Recall our procedure for solving optimization problems:

0. Go slowly and deliberately, talk to yourself, and maintain redundancy.
1. Understand the problem. (A picture may help.)
2. Think about your expectations of the solution.
3. Introduce variables if necessary.
4. Write an expression for what you want to optimize.
5. Eliminate all but one variable if necessary.
6. Find critical points (end points, singular points, stationary points)
7. Examine what happens at the critical points.
8. Check your answers, as you go, and in the end. Watch for consistent dimensions.

- Snell's Law. When light transitions between media with different speeds of light it travels so as to minimize the travel time between any two points on its path.
- This is why we can see each other.
- According to the wikipedia Snell's Law is attributed to Willebrord Snellius (1580-1626), but actually was first accurately described by Ibn Sahl at the Baghdad Court in 984. Sahl used the law to derive lens shapes that focus light with no aberrations.

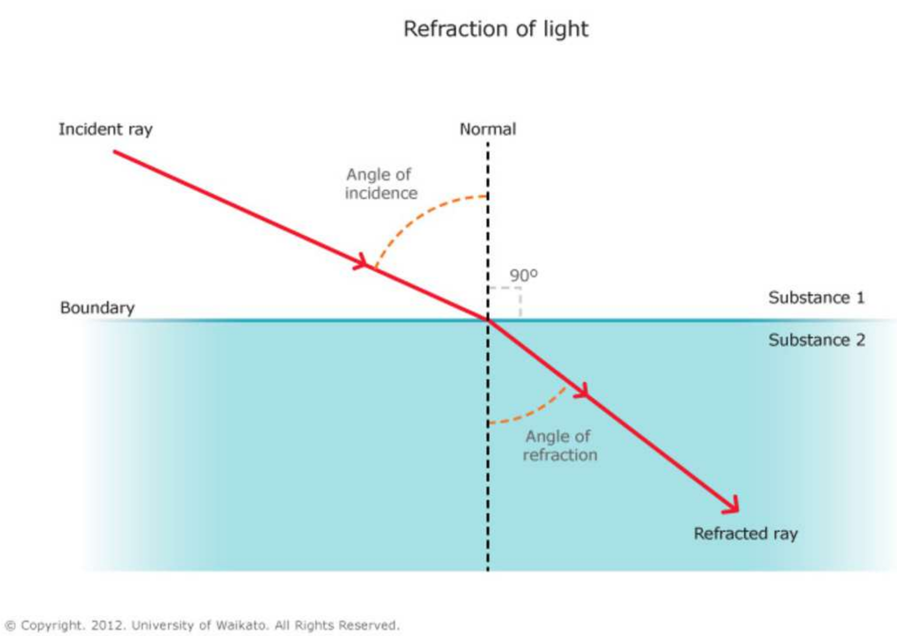


Figure 1. Refraction.

From https://www.sciencelearn.org.nz/images/49-refraction-of-light-in-water

Figure 2. Refraction Photo.

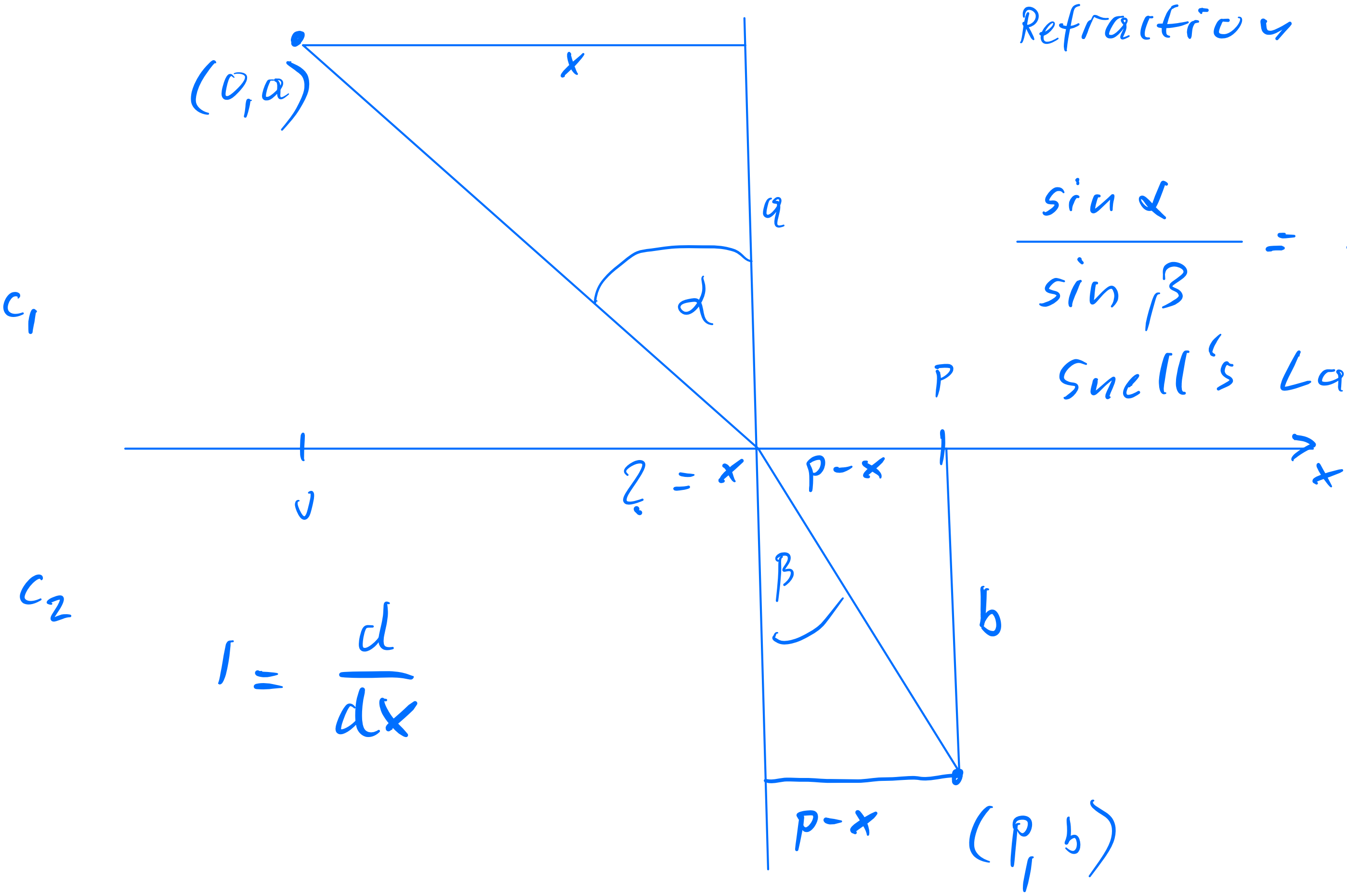
From https://en.wikipedia.org/wiki/Refractive_index

- How can we express this in terms of a formula?

Refraction

$$\frac{\sin\alpha}{\sin\beta} = \frac{c_1}{c_2}$$

Snell's Law



$$' = \frac{d}{dx}$$

T: travel time

$$T = \frac{\sqrt{a^2+x^2}}{c_1} + \frac{\sqrt{(p-x)^2+b^2}}{c_2}$$

$$0 = T' = \frac{1}{c_1}\cdot\frac{2x}{2\sqrt{a^2+x^2}} - \frac{1}{c_2}\frac{2(p-x)}{2\sqrt{(p-x)^2+b^2}}$$

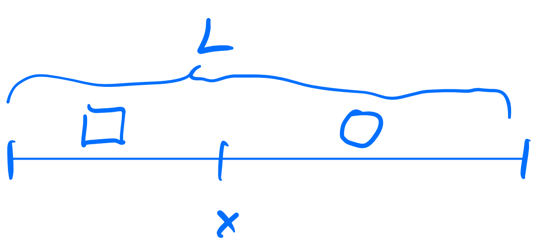
$$\frac{\sin\alpha}{c_1} - \frac{\sin\beta}{c_2} = 0$$

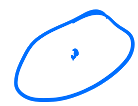
$$\frac{\sin\alpha}{c_1} = \frac{\sin\beta}{c_2} \quad \Big|\div \sin\beta\cdot c_1$$

$$\frac{\sin\alpha}{\sin\beta} = \frac{c_1}{c_2} \quad \text{Snell's Law}$$

$\frac{1}{c_1} \cdot \sin\alpha \qquad \frac{d}{dx}(p-x)^2 = -2(p-x)$

- Example: Problem 29, page 174. A wire of Length L is to be cut into two pieces. One piece is used to form a square, the other forms a circle. Where is the cut to be made so that the sum of the areas of these two shapes is
  - minimized,
  - maximized?
- Expectations?





$x < \frac{L}{2}$

$x = \frac{L}{2}$

$x > \frac{L}{2}$

$$A = A_\square + A_O = \begin{cases} \max & x = 0 \\ \min \end{cases}$$

$$A_\square = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$$

$$A_O = \pi r^2 \qquad r =$$

$$2\pi r = L - x$$

$$r = \frac{L-x}{2\pi}$$

$$= \pi \cdot \frac{(L-x)^2}{4\pi^2}$$

$$= \frac{(L-x)^2}{4\pi}$$

$$A = \frac{x^2}{16} + \frac{(L-x)^2}{4\pi} = \min$$

$$0 = A' = \frac{2x}{16} - \frac{2(L-x)}{4\pi} = 0$$

$$\frac{x}{8} - \frac{L-x}{2\pi} = 0 \quad \Big| \cdot 8\pi$$

$$\pi x - (L-x)4 = 0$$

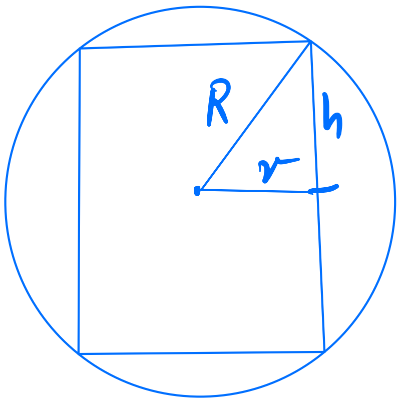
$$\pi x + 4x = 4L$$

$$x(\pi+4) = 4L$$

$$x = \frac{4}{\pi+4} L > \frac{L}{2}$$

- Example: What are the proportions of the largest cylinder that can be inscribed in a sphere? What's the ratio of its volume and that of the volume of the sphere?

$$R^2 - \frac{R^2}{3} = R^2\left(1 - \frac{1}{3}\right) = \frac{2}{3}R^2$$



$2h$

$$r^2 + h^2 = R^2$$
$$r^2 = R^2 - h^2$$

$$' = \frac{d}{dh}$$

$$V = 2\pi r^2 h = \max$$

$$= 2\pi(R^2 - h^2)h = \max$$

product rule

$$V' = 2\pi(R^2 - h^2 - 2hh) = 0$$

$$= 2\pi(R^2 - 3h^2) = 0$$

$$\frac{d}{dh}(R^2 - h^2) = -2h$$

$$1 + 3h^2$$

$$R^2 - 3h^2 = 0$$
$$3h^2 = R^2$$
$$h^2 = \frac{R^2}{3}$$

$$h = \frac{R}{\sqrt{3}}$$

$$h^2 = \frac{R^2}{3}$$

$$V = 2\pi(R^2 - h^2)\, h$$

$$= 2\pi\left(\frac{2}{3}R^2\right)\frac{R}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \cdot \underbrace{\frac{4\pi R^3}{3}}_{V_{sphere}}$$

$$\frac{V_{cyl}}{V_{sphere}} = \frac{1}{\sqrt{3}}$$

- If time permits: You want to build a square box holding a volume V. The material for the top is k times as expensive as the material for the bottom and the four sides. Let h be the height of the top and s the length of a side. What is the ratio h/s that minimizes the cost?