Math 1210-4

Notes of 10/16-17/2017

Chapter 2 Summary

• Exam 2 in our class will take place on Wednesday, October 18, 2017. It will have 10 questions and cover chapter 2 of our textbook. 7 of the questions are differentiation problems. There will also be a simple differentials problem and two simple related rates problems.

• Several questions ask for a polynomial in standard form. Recall that this means you collect terms and write them by decreasing degree. For example, expanding and collecting terms in

\[ p(x) = (x^2 + 2x + 3)(x - 1) \]

\[ p(x) = (x^2 + 2x + 3)(x - 1) \]
\[ = x^3 + 2x^2 + 3x - (x^2 + 2x + 3) \quad (1) \]
\[ = x^3 + x^2 + x - 3. \]

The last expression is the standard form of \( p \).

The Subject

• The following list is neither complete nor self contained. Rather it is meant to trigger your memory and activate your comprehension. If
any of these points are not clear to you make sure you review the relevant material before the exam. You want to understand everything that’s indicated here but not everything will be covered by the exam.

- Useful formulas from your past include

\[
\begin{align*}
\sin^2 x + \cos^2 x &= 1 & \text{by definition} \\
V &= \frac{4\pi r^3}{3} & \text{volume of a sphere} \\
A_S &= 4\pi r^2 & \text{surface area of a sphere} \\
A_C &= \pi r^2 & \text{area of a circle}
\end{align*}
\]
You want to have a thorough grasp of the **concept of a derivative**:

- It’s the slope of the tangent.
- It’s the limit of the slopes of the secants.
- It measures how rapidly a function is changing.
- The derivative of location is velocity
- The derivative of velocity is acceleration
- The derivative is defined as

\[
 f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}
\]  

- There are a large number of **notations**, including the following. Suppose that \( y = f(x) \).

\[
 f'(x) = \frac{dy}{dx} = \frac{d}{dx}f(x) = Dy = Df(x) = D_x f(x) = D_x y.
\]  

You should be familiar, and in fact comfortable, with all of those notations, and be able to use whichever is most appropriate or convenient for a given problem.
• To compute derivatives we apply their properties, i.e., differentiation rules shown in the box on this page. You don’t want to memorize these formulas! Instead you want to use them so often that you can’t possibly forget them, and you want to be able to derive them and explain them to your friends.

\[
\frac{d}{dx} x^r = rx^{r-1} \quad \text{Power Rule}
\]
\[
(f + g)' = f' + g' \quad \text{Sum Rule}
\]
\[
(f - g)' = f' - g' \quad \text{Difference Rule}
\]
\[
(kf)' = kf' \quad \text{Constant Multiple Rule}
\]
\[
\frac{d}{dx} \sin x = \cos x \quad \text{Sine Rule}
\]
\[
\frac{d}{dx} \cos x = -\sin x \quad \text{Cosine Rule}
\]
\[
(uv)' = u'v + uv' \quad \text{Product Rule}
\]
\[
\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \quad \text{Quotient Rule}
\]
\[
\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \quad \text{Chain Rule}
\]

\[
(kf)' = kf' \quad \text{(4)}
\]

• The sum and constant multiple rules together mean that differentiation is linear.
• You should be able to apply these rules in combination, and differentiate anything that can in fact be differentiated by these rules.

• Apply the onion method: Ask what is the nature of the outermost layer? Specifically, what is the last operation when evaluating a given expression? Then apply the appropriate rule, and proceed similarly to differentiate subexpressions.

• Example:
Differentiation can be repeated, giving rise to higher derivatives, for example:

\( f(x) = 2x^5 + 3x^4 - 4x^3 + 5x^2 - 6x + 7 \)
\( f'(x) = 10x^4 + 12x^3 - 12x^2 + 10x - 6 \)
\( f''(x) = 40x^3 + 36x^2 - 24x + 10 \)
\( f'''(x) = 120x^2 + 72x - 24 \)
\( f^{(4)}(x) = 240x + 72 \)
\( f^{(5)}(x) = 240 \)  

(6)

Note how each differentiation reduces the degree of the polynomial by 1.

In general, a function \( f \) is a polynomial of degree up to \( n \) if and only if the \((n + 1)\)-th derivative of \( f \) is everywhere zero.

Differentiation can be done implicitly. For example, thinking of \( y \) as a function of \( x \), we get

\[ x^2 + y^2 = 1 \quad \Rightarrow \quad 2x + 2yy' = 0 \quad \Rightarrow \quad y' = -\frac{x}{y} \]  

(7)

This process is called **implicit differentiation**.

Implicit differentiation occurs frequently in **Related Rates Problems**: Understand the problem, introduce variables, write one or more equations that hold at all time, differentiate, obtain equations that involve rates (derivatives), solve for what you want to know, make
sure the dimensions in your results are consistent, substitute numbers only at the end if at all.

• **Differentials:** The change in a function value is approximately equal to the change in the independent variable, multiplied with the derivative. This is expressed as

\[
\Delta y \approx dy = f'(x)dx = f'(x)\Delta x \quad (8)
\]

where in this context \(dx\) and \(dy\) are variables called *differentials* and \(\Delta x\) and \(\Delta y\) are the corresponding changes in \(x\) and \(y\).

• **Linear Approximation.** The linear approximation of a function \(f\) at a point \((x_0, f(x_0))\), say, is simply the tangent. It is given by

\[
T(x) = f(x_0) + f'(x_0)(x - x_0) \quad (9)
\]

• In a profound sense (which, however, is beyond our scope) the approximation by the tangent is the better the closer \(x\) is to \(x_0\).

• Don’t confuse a differentiation rule such as the product rule

\[
\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad (10)
\]

with the corresponding differential statement

\[
d(uv) = udv + vdu. \quad (11)
\]
Major error sources in differentiation:

- Failure to appreciate with respect to which variable (the independent variable) you differentiate, and which other variables depend on the independent variable.

- Misapplication of the product or quotient rules: The derivative of a product (or quotient) does not equal the product (or quotient) of the derivatives.

- Misapplication of the chain rule. You need to be clear on the sequence in which you apply functions. For example:

  \[
  \frac{d}{dx} f(g(x)) = f'(g(x)) g'(x)
  \]

  \[
  \sin^2 x = (\sin x)^2 \quad \frac{d}{dx} \sin^2 x = 2 \sin x \cos x
  \]

  \[
  \sin x^2 = \sin(x^2) \quad \frac{d}{dx} \sin x^2 = 2x \cos x^2
  \]

- Algebraic errors. These occur in particular when simplifying an expression, for example in preparation for computing a higher order derivative.
Notation and Writing

• Unlike in WeBWorK which will accept a correct answer in any form, write polynomials in factored form, or in standard form. The standard form of a polynomial of degree $n$ is, of course:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0. \quad (13)$$

• Here are some suggestions for writing your work and answers:

  • Use fractions instead of decimals.
  • Avoid mixed numbers
  • Cancel common factors in numerator and denominator.
  • Unfactor a polynomial only if you have a good reason to do so. (For example, when you want to combine it with another polynomial.)
  • It’s OK to have a radical in the denominator.
  • Use equals signs, for example when solving an equation, or when simplifying an expression.

  • **Most important: write clearly, and explain what you are doing!**
11/8/17: Calculus Carnival

This week's Lab: due to popular demand. Discussion of Exam (experimental)

altitude increasing \( h' \) \( \frac{1}{2} \) given
Area increasing \( A' \) \( \frac{1}{2} \) \( \frac{d}{dt} \)

\[
A = \frac{1}{2} bh
\]
\[
b' = \frac{d}{dt}
\]

\[
A' = \frac{1}{2} (bh)'
\]
\[
A' = \frac{1}{2} (b'h + bh')
\]

\[
2 A' = b'h + bh'
\]
\[
2 A' - bh' = b'h
\]

\[
b' = \frac{2 A' - bh'}{h}
\]

substitute \#5

\[
f^2
\]
\[ \frac{f}{s} \rightarrow \frac{1}{s} \cdot f \rightarrow \frac{1}{s} \rightarrow \frac{e^{-st}}{s} \rightarrow f \frac{e^{-st}}{s} \]

\[
\frac{d}{dx} \left( x + \frac{1}{z} \right) = \frac{d}{dx} x + \frac{d}{dx} \frac{1}{z} = 1 + 0 = 1
\]

\[
\frac{d}{dx} \left( \frac{1}{z} \cdot x \right) = \frac{1}{z} \frac{d}{dx} x = \frac{1}{z}
\]

\[
\frac{d}{dx} (k \cdot f(x)) = k \cdot \frac{d}{dx} f(x)
\]

\[
\frac{d}{dx} \sin(2x) = 2 \cos(2x) = 2 \cos 2x
\]

\[
\sqrt{9.9^2} \approx \sqrt{4^2} + 0.9 \cdot \frac{1}{2 \cdot 4^2} = 2 + \frac{0.9}{4} = 2 + \frac{9}{40}
\]

\[
= \frac{89}{40}
\]
\[ dV = \frac{d}{dr} V \cdot dr = h \text{ const.} \]

\[ (dV = \int_a^b f(x) \, dx) \]

\[ V = \pi r^2 h \]

\[ \frac{dV}{dr} = \pi h \frac{d}{dr} r^2 = 2\pi rh \]

\[ dV = 2\pi rh \, dr \]

\[ \frac{dV}{V} = \frac{2\pi rh}{\pi r^2 h} \frac{dr}{r} \]

\[ \frac{dV}{V} = \frac{2 \, dr}{r} \]

\[ Y = f(x) \]

\[ f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \]

\[ \Delta y \approx f'(x) \Delta x \]
\[ V = V(h) \]

\[
\frac{dV}{dh} = \frac{d}{dh} \pi r^2 h = \pi r^2
\]

\[
dV = \pi r^2 dh
\]

\[
\frac{dV}{V} = \frac{\pi r^2 \, dh}{\pi r^2 \, h} = \frac{dh}{h}
\]

\[ d = \sqrt{v^2 + u^2} \]
approximate 12.6

\[ f(x) = x^{\frac{2}{3}} \]

\[ f(125) = 125^{\frac{2}{3}} = \left(\left(125\right)^{\frac{1}{3}}\right)^2 = 25 \]

\[ f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \Delta x \]

\[ f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \]

\[ f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \]

\[ f(x) = x^{\frac{2}{3}} \quad x_0 = 125 \quad f(x_0) = 25 \]

\[ f'(x) = \frac{2}{3} x^{-\frac{1}{3}} \quad f'(x_0) = \frac{2}{3} \cdot 125^{-\frac{1}{3}} \]

\[ = \frac{2}{3} \cdot \frac{1}{5} = \frac{2}{15} \]

\[ 12.6^{\frac{1}{3}} = f(12.6) \approx f(x_0) + \Delta x \cdot f'(x_0) \]

\[ = 25 + 1 \cdot \frac{2}{15} = 25 + \frac{2}{15} \]
\[ d^2 + h^2 = s^2 \]

\[ 2dd' + \dot{0} = 2ss' \]

\[ s' = \frac{2dd'}{2s} = \frac{dd'}{s} \]

\[ d = \sqrt{s^2 - h^2} \]
\[ V = \frac{\pi R^2 h}{3} = \frac{\pi R^2 h^3}{3H^2} \]

\[ \frac{r}{h} = \frac{R}{H} \]

\[ T = \frac{R}{H} h \]

**no leakage**

\[ V' = \frac{\pi R^2}{3H^2} \frac{dh}{dt} h^3 = \frac{\pi R^2}{3H^2} \frac{3h^2 h'}{H^2} \]

\[ = \frac{\pi R^2}{H^2} h^2 h' \]

**pump rate** \( = \frac{\pi R^2}{H^2} h^2 h' + \text{leak rate} \)
\( x^2 + xy + y^2 = 3 \)

\( x = 1 \quad y = 1 \)

\( Y = Y(x) \)

\( \text{Diff. w.r.t. } x \)

\[ 2x + y + xy' + 2yy' = 0 \]

\[ xy' + 2yy' = -2x - y \]

\[ y'(x + 2y) = -2x - y \]

\[ \frac{y'}{x + 2y} = \frac{-2x - y}{x + 2y} \]

\( x = y = 1 \)

\[ \frac{y - 1}{x - 1} = -1 \]

\[ y - 1 = -1(x - 1) \]

\[ y - 1 = 1 - x \]
\[ y = 1 - x + 1 = -x + 2 \]

\[ \frac{y - y_0}{x - x_0} = m \]

\[ y = -x + 2 \]