Math 1210-23

Notes of 2/23/24

3.3 Local and Global Extreme Values

- The concepts are more easily explained in terms of a picture:

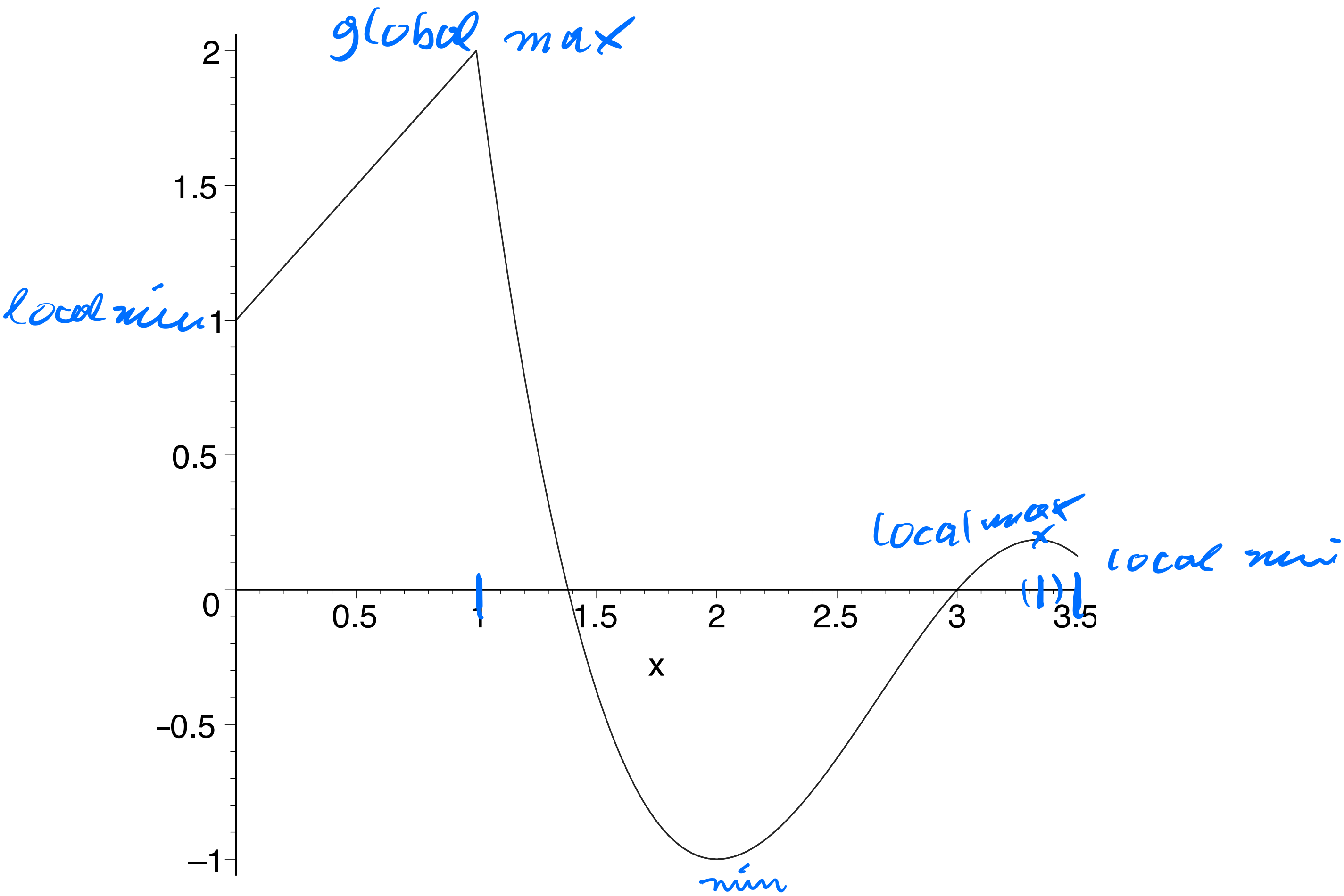
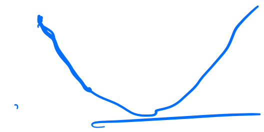


Figure 1. Extreme Values.

- Definition: Let S be the domain of f. Assume S is an interval. $f(c)$ is a **local maximum value** if there is an interval (a, b) containing c such that $f(c) \geq f(x)$ for all x in $(a, b) \cap S$ (the intersection of (a, b) and S).
- Similarly we define **local minimum value** (exercise).
- Exercise: Think about how this defintion works at the endpoints of S.
- As before we define $f(c)$ as a **global maximum value** if $f(c) \geq x$ for all x in S.
- Similarly we define **global minimum value** (exercise).
- Collectively we refer to local or global minimum or maximum values as local or global **extreme values**.
- Of course, every global extreme value is also a local extreme value.
- In the literature, instead of "local" or "global" you might see the words "relative" or "absolute", respectively.

- We saw last week that extreme values can occur only at **critical points** of which there are three kinds:
  - **End Points** of intervals,
  - **Singular Points** where the derivative does not exist, and
  - **Stationary Points** where the derivative is zero.
- The most frequently relevant kind of critical points are stationary points.
- Note that the word "point" refers to a value of the independent variable, and the word "value" refers to a value of the dependent variable.
- The main approach to solving minimization and maximization problems consists of computing all critical points and then figuring out what happens at those points.
- How do we figure this out. Often things are clear from the context.
- But another set of answers is provided by the first and second derivative tests.

The First Derivative Test



- Think of moving from left to right, i.e., x increases.
- If at a stationary point c the derivative changes sign from being positive to being negative then $f(c)$ is a local maximum value.
- If at a stationary point c the derivative changes sign from being negative to being positive then $f(c)$ is a local minimum value.

The Second Derivative Test

- If at a stationary point c the second derivative is positive then $f(c)$ is a local minimum value.
- If at a stationary point c the second derivative is negative then $f(c)$ is a local maximum value.

Restatements

- $f(c)$ is a local $\begin{cases} \text{minimum} \\ \text{maximum} \end{cases}$ value if $f'(x)$ changes sign at c from $\begin{cases} \text{negative} \\ \text{positive} \end{cases}$ to $\begin{cases} \text{positive} \\ \text{negative} \end{cases}$.

- The first, $-$ve to $+$ve, is of course true in particular, if $f''(c) > 0$, and the second is true if $f''(c) < 0$.

- This gives rise to the second derivative test:

  $f(c)$ is a local $\begin{cases} \text{minimum} \\ \text{maximum} \end{cases}$ value if $f'(c) = 0$

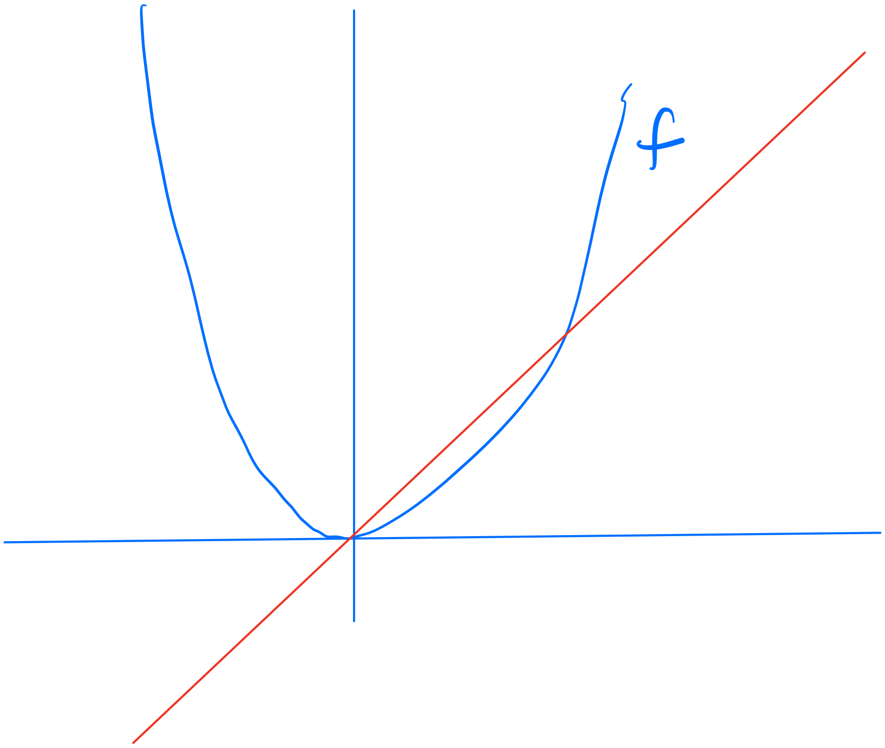
and $\begin{cases} f''(c) > 0 \\ f''(c) < 0 \end{cases}$.

- Example

$$f(x) = x^2$$

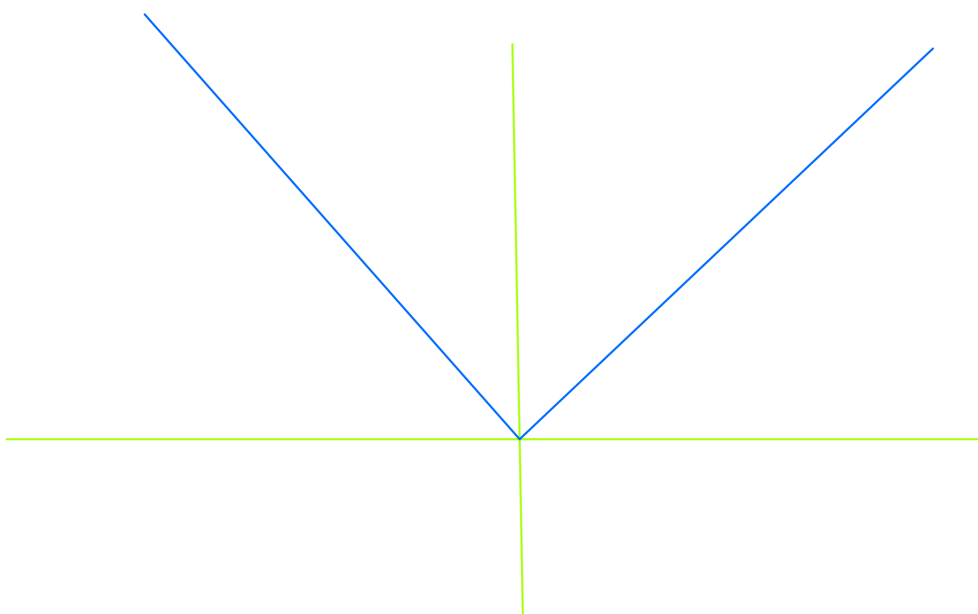
$f'(x) = 2x$

$f''(x) = 2 > 0$



- Example

$$f(x) = |x|$$



- Example 2: $-2 \le x \le 4$ and

$$\begin{aligned} f(x) &= \frac{1}{3}x^3 - x^2 - 3x + 4 \\ f'(x) &= x^2 - 2x - 3 \\ &= (x-3)(x+1) \\ f''(x) &= 2x - 2 \\ &= 2(x-1) \end{aligned}$$

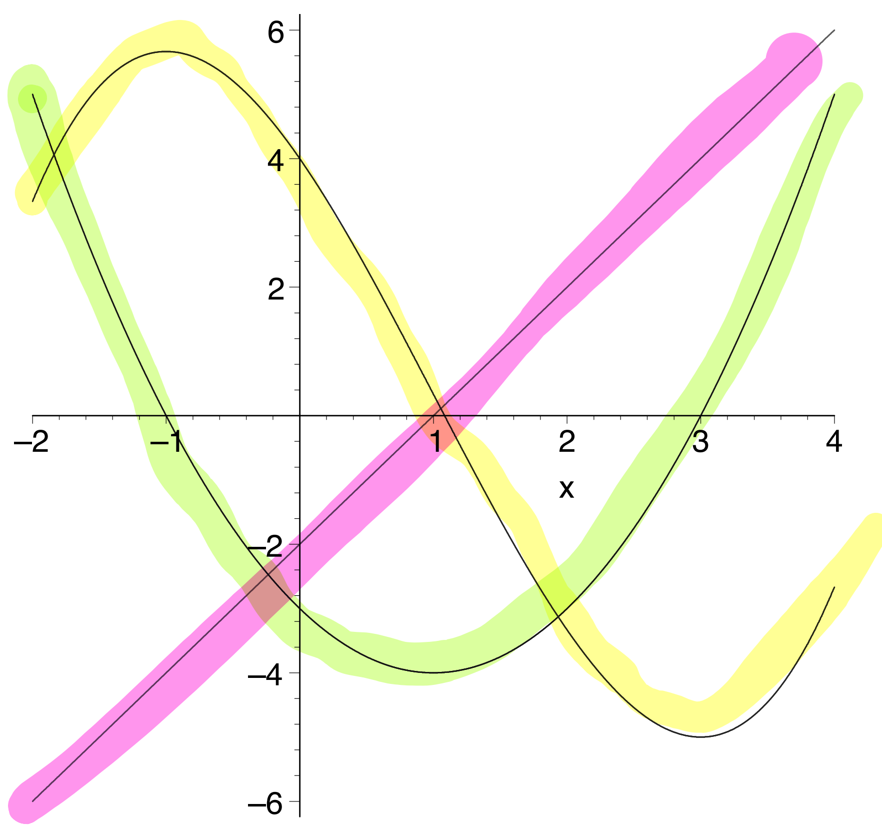
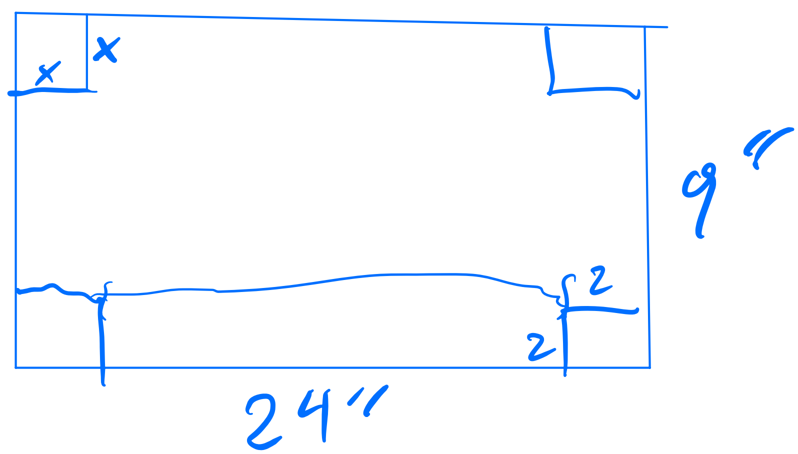


Figure 2. $f(x) = \frac{x^3}{3} - x^2 - 3x + 4$.

- The fun begins. Entering Section 3.4, Word Problems. We will spend the rest of today, and Monday and Tuesday of next week on word problems.

- Example 1: A rectangular box is to be made from a piece of cardboard 24 inches long and 9 inches wide by cutting out identical squares from the four corners and turning up the sides. Find the dimensions of the box with the maximum possible volume. What is that volume?

$x = 2$



$$0 \le x \le \frac{9}{2}$$

$x=2 \quad V = 20 \cdot 5 \cdot 2 = 200 \text{ in}^3$

$$V = (24-2x)(9-2x)x$$
$$= (216 - 48x - 18x + 4x^2)x$$
$$= 216x - 66x^2 + 4x^3$$
$$V' = 216 - 132x + 12x^2 = 0$$

$$= 12(18 - 11x + x^2)$$

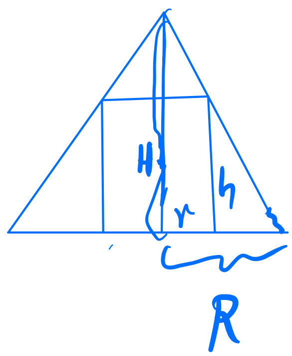
$$= 12(x-2)(x-9) = 0$$

$$x = 2$$

or $\cancel{x = 9}$

- Example 3: Find the dimensions of the largest right circular cylinder that can be inscribed in a given right circular cone. What fraction of the volume of the cone is taken up by the volume of the cylinder?

- See Figure 4 on page 168.



$$V_{cone} = \frac{\pi R^2 H}{3}$$

$$V = V_{cylinder} = \pi r^2 h$$

$$\frac{R}{H} = \frac{r}{H-h}$$

$$r = \frac{R}{H}(H-h)$$

$$V = \pi r^2 h = \pi \frac{R^2}{H^2}(H-h)^2 h = \min$$

$$0 \le h \le H$$

$$V' = \pi \frac{R^2}{H^2}\left(-2(H-h)h + (H-h)^2\right) = 0$$

$$= \pi \frac{R^2}{H^2}(H-h)\left(-2h+H-h\right)$$

$$= \pi \frac{R^2}{H^2}(H-h)(H-3h)$$

$\cancel{h=H}$ $\qquad h = \frac{H}{3}$

$$h = \frac{H}{3} \qquad H - \frac{H}{3} = H\left(1-\frac{1}{3}\right) = \frac{2}{3}H$$

$$r = \frac{R}{H}(H-h) = \frac{R}{H}\left(H - \frac{H}{3}\right)$$

$$= \frac{2}{3}H \cdot \frac{R}{H}$$

$$= \frac{2}{3}R$$

$$V = \pi r^2 h = \pi \frac{4}{9}R^2 \cdot \frac{H}{3}$$

$$= \pi \frac{4}{27} R^2 H$$

$$V_{cone} = \frac{1}{3}\pi R^2 H$$

$$\frac{V}{V_{cone}} = \frac{\pi \frac{4}{27} R^2 H}{\frac{1}{3}\pi R^2 H} = \frac{4/27}{1/3} = \frac{4}{9}$$

$$\frac{4/27}{1/3} = 3 \cdot \frac{4}{27} = \frac{4}{9}$$