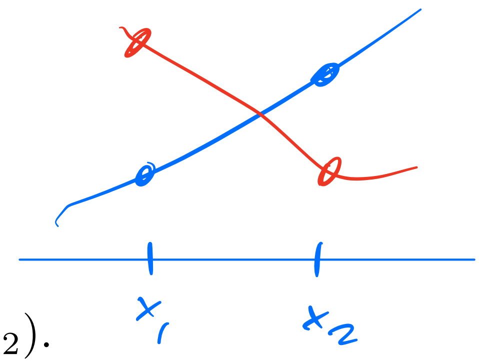
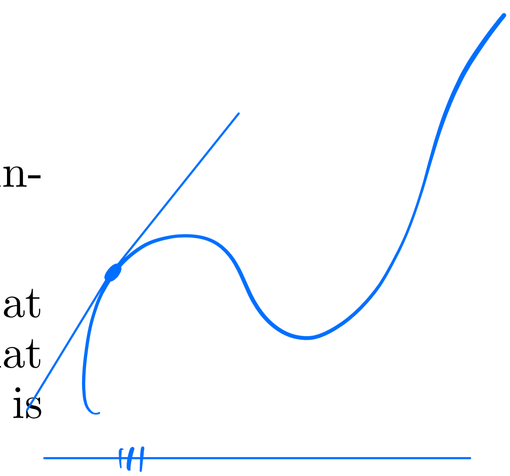
Math 1210-23

Notes of 02/21/24

- We are in the coolest part of Calculus: Minimization and Maximization (Optimization)
- Before looking more deeply into this let's look more closely at the connection between derivatives and the shape of a graph.



- Suppose f is defined on an interval I.
- Definition:
  - f is **increasing** on I if
  $$x_1 < x_2 \qquad \Longrightarrow \qquad f(x_1) < f(x_2).$$
  - f is **decreasing** on I if
  $$x_1 < x_2 \qquad \Longrightarrow \qquad f(x_1) > f(x_2).$$
  - f is **strictly monotonic** on I if it is increasing on I or it is decreasing on I.



- We say that f is increasing or decreasing at a point x if there exists an interval I that contains x in its interior and on which f is increasing or decreasing, respectively.
- In other texts you might see "$\leq$" and "$\geq$" instead of "$<$" and "$>$".
- To emphasize that equality is excluded we also say that f is **strictly increasing** or **strictly decreasing** on I.

- Monotonicity is closely related to the derivative being positive or negative:

$$f'(x) > 0 \quad \text{for all } x \text{ in } I \qquad \Longrightarrow \qquad f \text{ is increasing on } I$$

$$f'(x) < 0 \quad \text{for all } x \text{ in } I \qquad \Longrightarrow \qquad f \text{ is decreasing on } I$$

- This makes geometric sense.
- A proof is in section 3.6.

- Example 1.

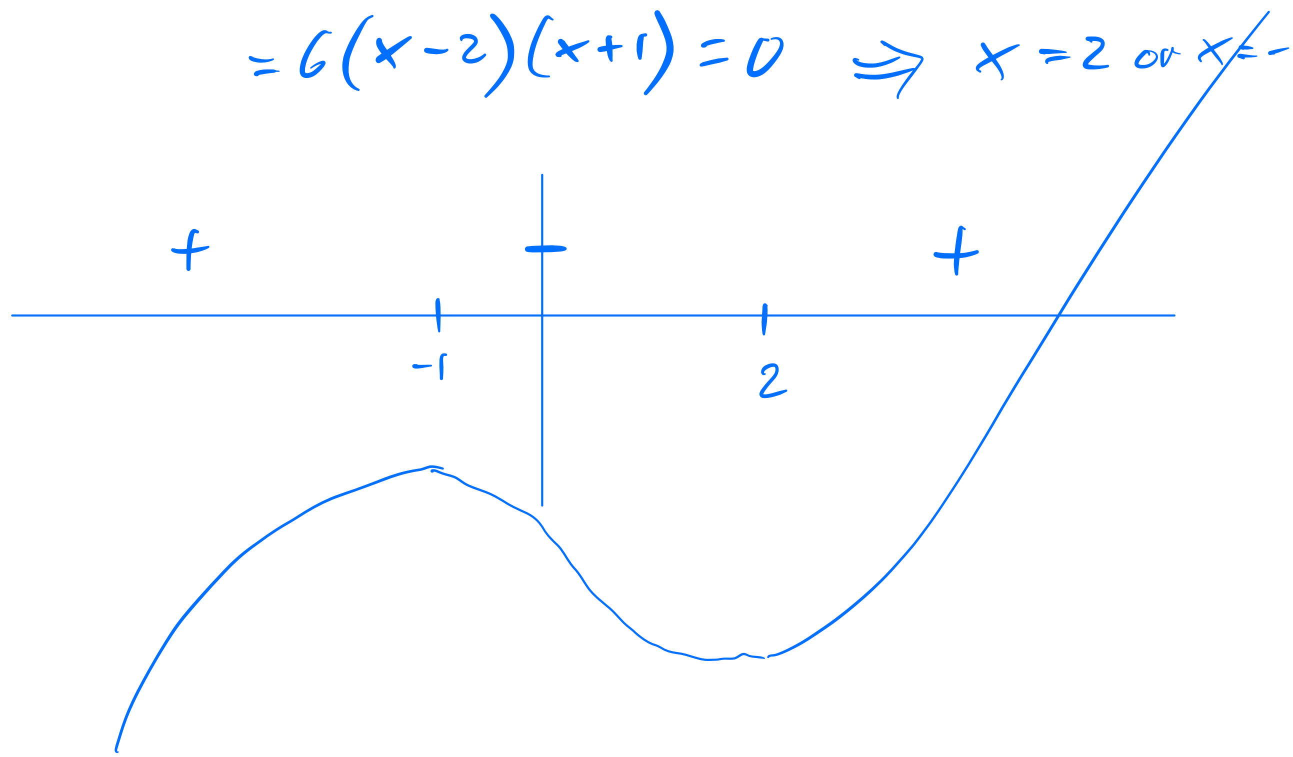
$$f(x) = 2x^3 - 3x^2 - 12x + 7$$

inc?
dec?

$$f'(x) = 6x^2 - 6x - 12$$
$$= 6(x^2 - x - 2)$$
$$= 6(x-2)(x+1) = 0 \Rightarrow x = 2 \text{ or } x = -1$$

$$f'(0) = -12$$

f':



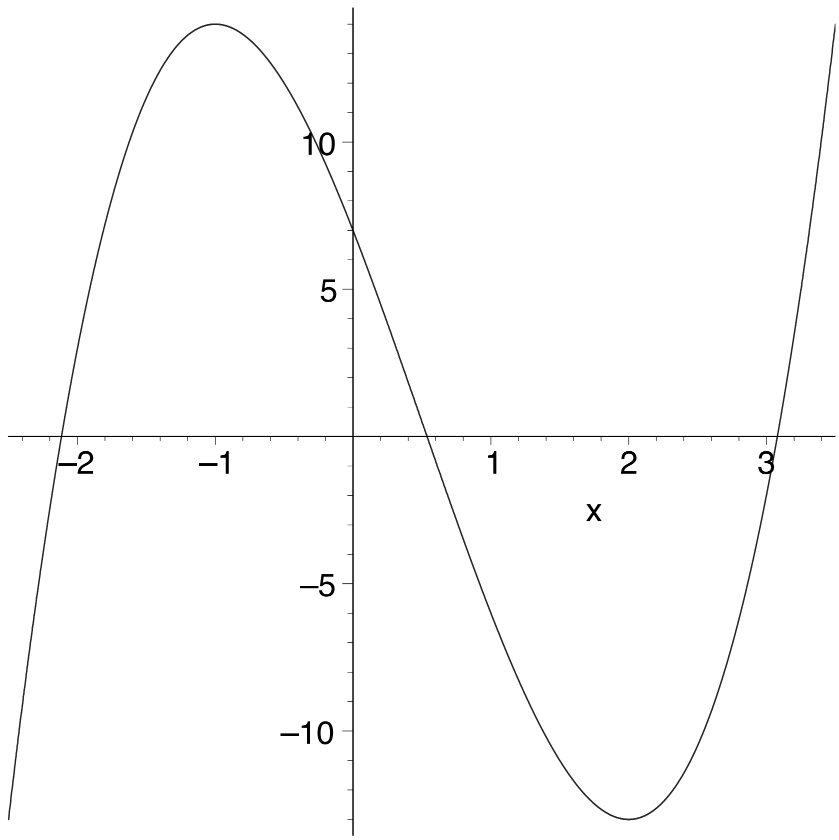


Figure 1. Example 1: $f(x) = 2x^3 - 3x^2 - 12x + 7$.

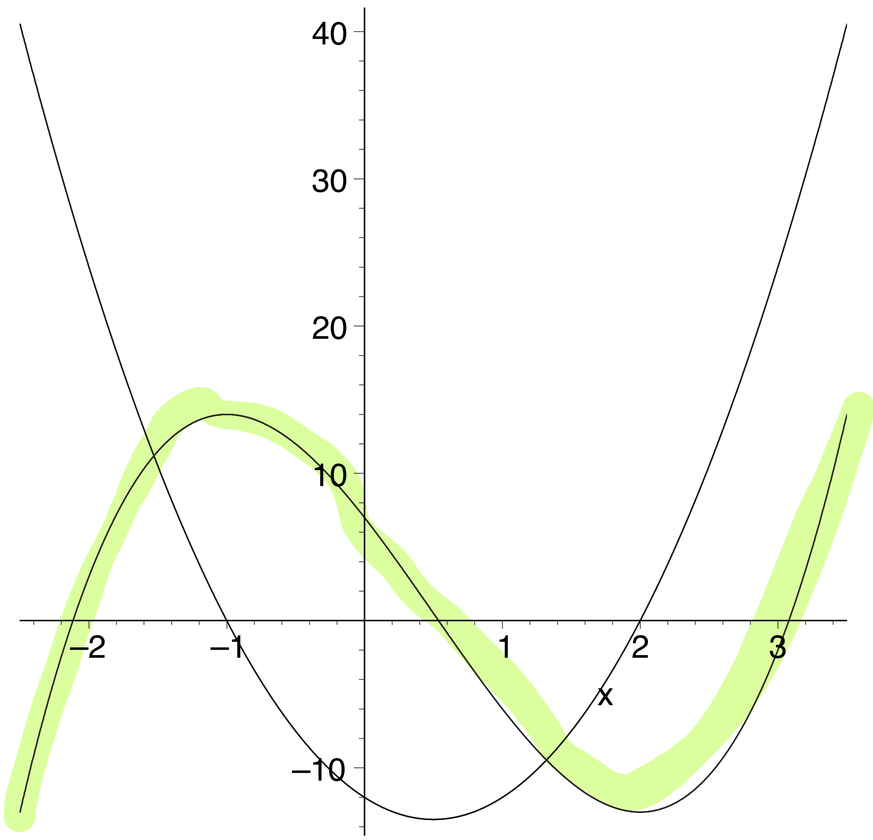


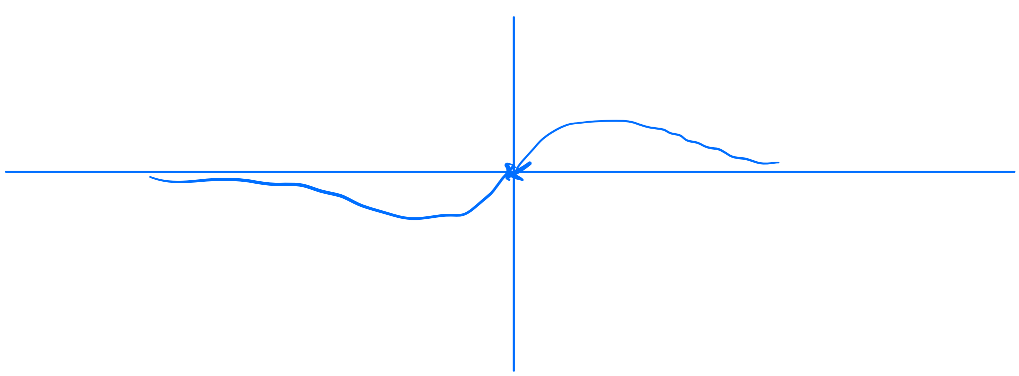
Figure 2. Example 1: f and its derivative.

- Example 2:

$$g(x) = \frac{x}{1+x^2}$$

$$g(-x) = \frac{-x}{1+(-x)^2} = -\frac{x}{1+x^2} = -g(x)$$

- Already briefly looked at this on Wednesday.



$$g'(x) = \frac{1+x^2 - x \cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

$g'(x) = 0 \quad x = \pm 1.$

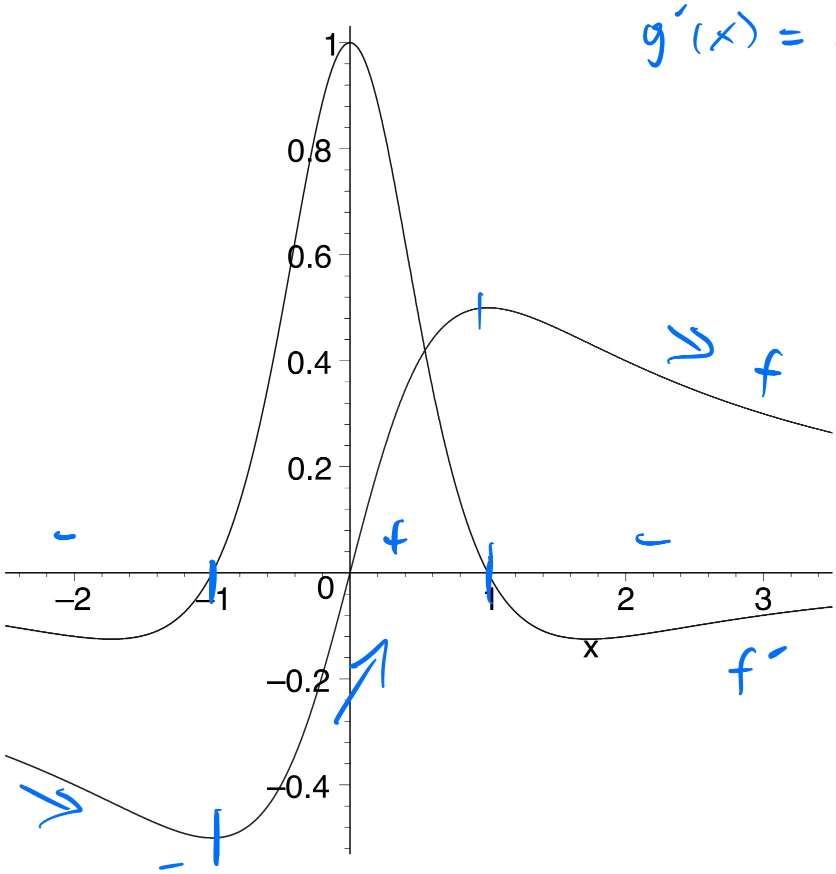
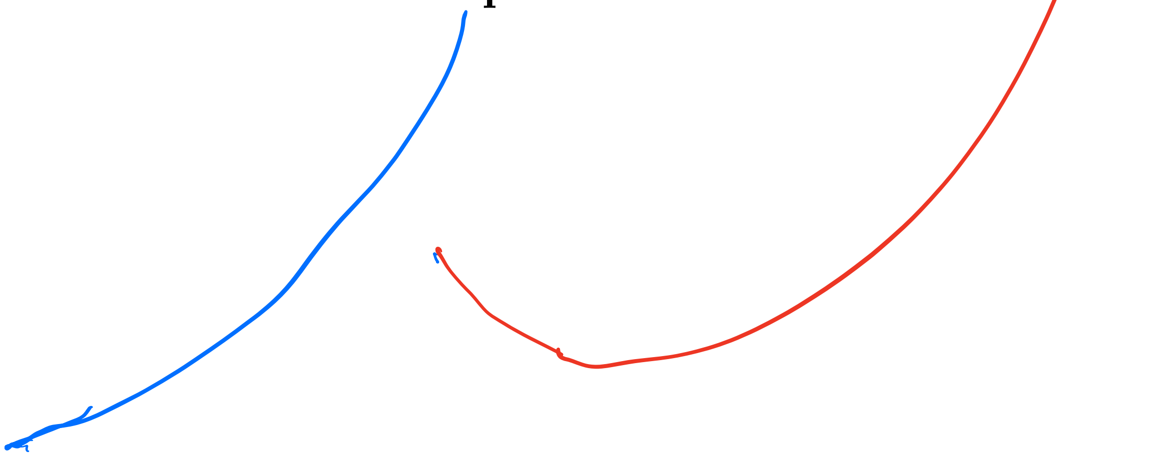


Figure 3. Example 2: g and its derivative.

Second Derivatives

- If the second derivative is positive the first derivative is increasing. The function itself is getting steeper as we go up—we are making a left turn as we travel from left to right. We say the function is **concave up**.



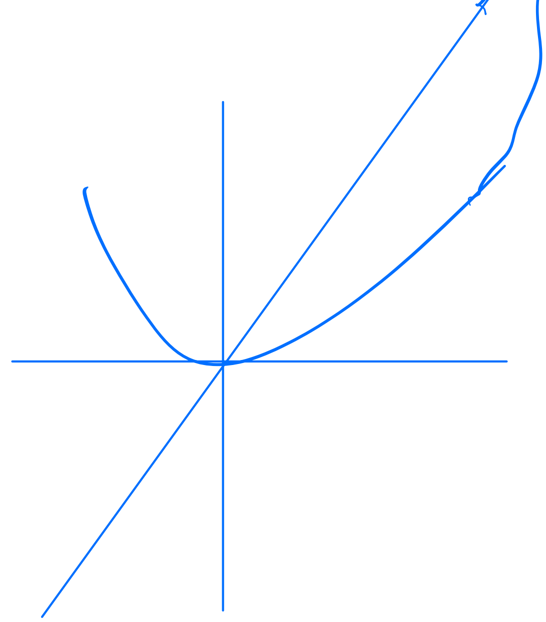
- If the second derivative is negative the first derivative is decreasing. The function itself is getting less steep as we go up—we are making a right turn as we travel from left to right. We say the function is **concave down**.

- Example:

$$f(x) = x^2$$

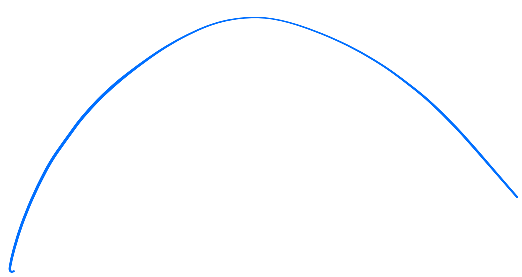
$f'(x) = 2x$

$f''(x) = 2 >$

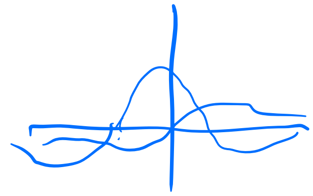


- Example

$$f(x) = -x^2$$



- Back to example 2:



$$g(x) = \frac{x}{1+x^2}$$

$$g'(x) = \frac{1-x^2}{(1+x^2)^2}$$

$$g''(x) = \frac{2x(-3+x^2)}{(1+x^2)^3}$$

$$g''(x) = \frac{-2x(1+x^2)^2 - (1-x^2)\,2(1+x^2)\,2x}{(1+x^2)^4}$$

$$= \frac{-2x(1+x^2) - (1-x^2)\cdot 4x}{(1+x^2)^3}$$

$$= \frac{2x(-1-x^2-2+2x^2)}{(1+x^2)^3}$$

$$= \frac{2x(-3+x^2)}{(1+x^2)^3}$$

$g''(x) = 0$

$x = 0 \qquad x^2 = 3 \qquad x = \pm\sqrt{3}$

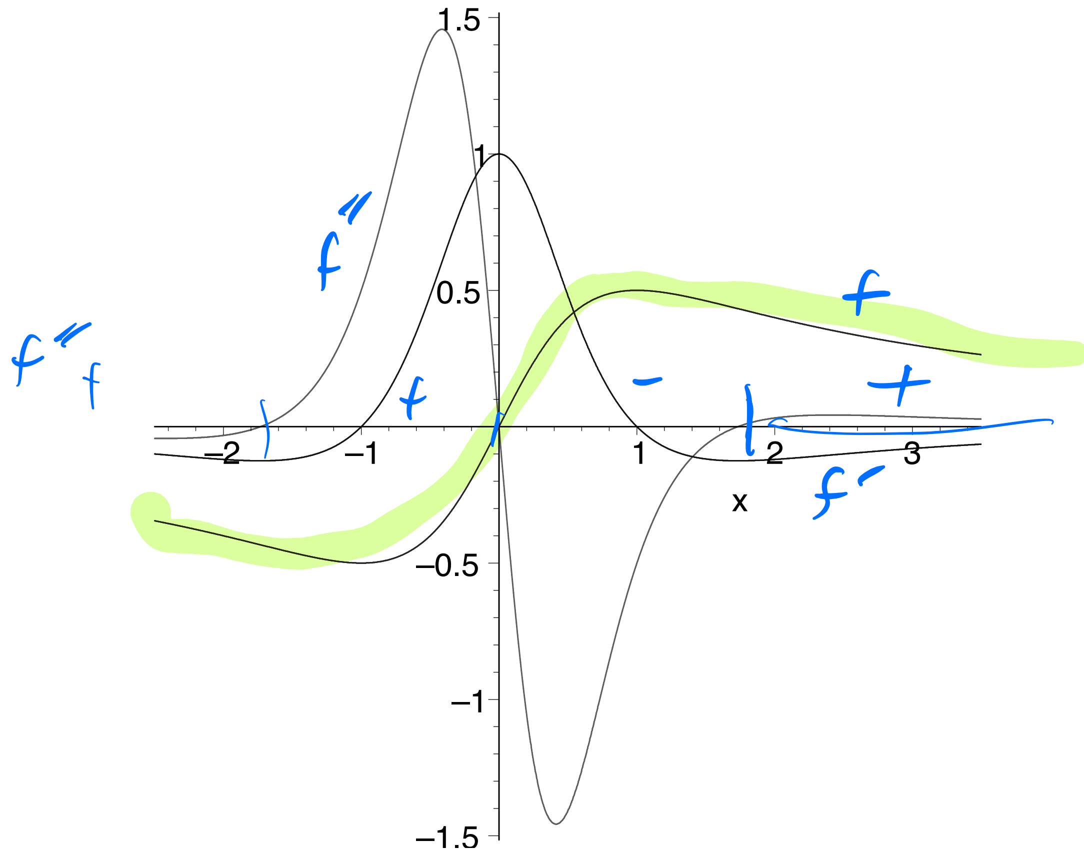


Figure 4. Example 2: g and its first and second derivative.

interval	sign g"	Concavity
$x < -\sqrt{3}$	< 0	c.d.
$-\sqrt{3} < x > 0$	> 0	c.u.
$0 < x < \sqrt{3}$	< 0	xc.d.
$x > \sqrt{3}$	> 0	c.u.

- A point $\big(c, f(c)\big)$ is an **inflection point** if the function changes from concave up to concave down at $x = c$.

- Note that

$$f\left(\pm\sqrt{3}\right) = \frac{\pm\sqrt{3}}{1+3} = \frac{\pm\sqrt{3}}{4}.$$

- Our function g has the inflection points

$$\left(-\sqrt{3}, \frac{-\sqrt{3}}{4}\right), (0,0), \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$$

- An inflection point $\big(c, f(c)\big)$ is a **saddle point** if $f'(c) = 0$.
- Example:

$$f(x) = x^3$$

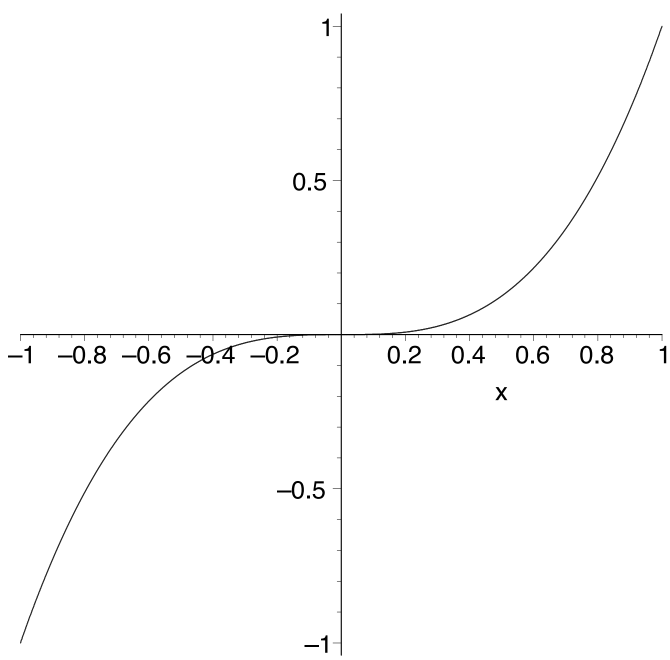


Figure 5. Graph of $f(x) = x^3$.

Note that a "critical point" is a value of the independent variable, but an "inflection point" is a point on the graph of the function.

Summary of last two meetings

- Extreme values occur only at critical points of which there are three kinds:
  - end points of intervals (boundary points)
  - singular points (where the derivative does not exist)
  - stationary points (where the derivative is zero)
- The signs of the first two derivatives tell us about the geometry of the graph of f.
  - $f'(x) < 0$: f is decreasing at x.
  - $f'(x) > 0$: f is increasing at x.
  - $f'(x) = 0$: f has a horizontal tangent at x.
  - $f''(x) < 0$: f is concave down at x.
  - $f''(x) > 0$: f is concave up at x.
  - $f''(x) = 0$ and changes sign at x: f has an inflection point at x.
  - $f''(x) = 0$, $f'(x) = 0$, and f'' changes sign at x: f has a saddle point at x.

Do not try to memorize these facts. They should be obvious from the meaning of the derivative. If they are not think about the issues until they become clear.