Math 1210-23

Notes of 02/20/24

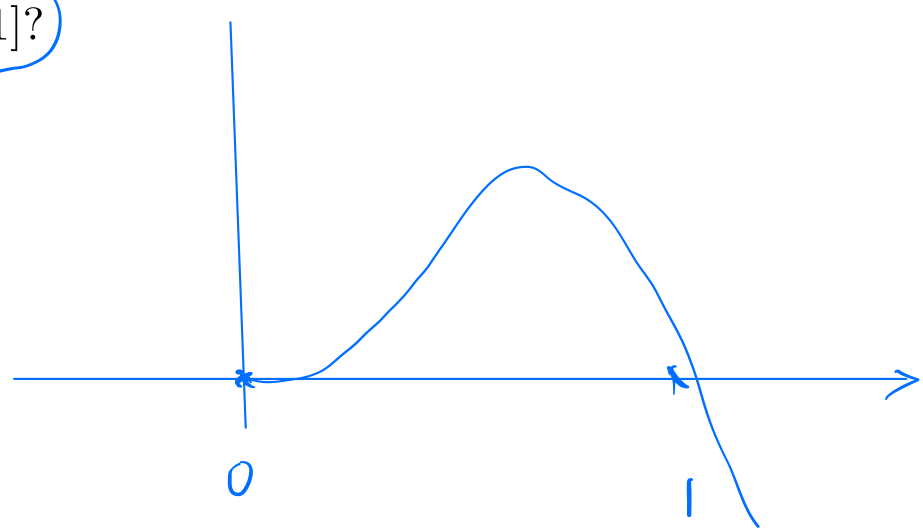
- We are done with Chapter 2. Exam 2 will cover Chapter 2, and will take place next week, after the last hw (hw 7) covering chapter 2 is closed.

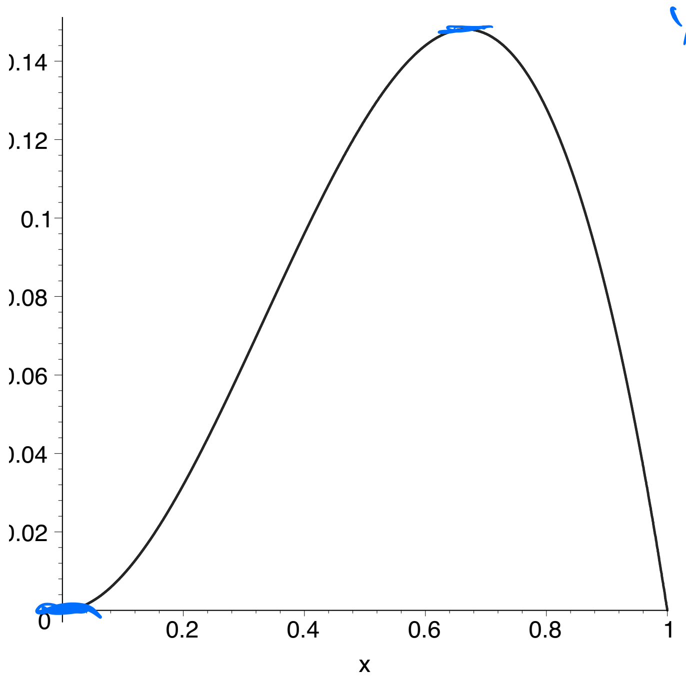
3.1 Minima and Maxima

- Entering new chapter
- Minimization and Maximization: obviously important, and major application of Calculus.
- Example: Recall an old home work problem: What's the maximum value of

$$f(x) = x^2(1-x)$$

in the interval $[0, 1]$?





$y \approx 0.145$

Figure 1. Graph of $f(x) = x^2(1-x)$.

$$f(x) = x^2(1-x) = x^2 - x^3$$

$$f'(x) = 2x - 3x^2$$
$$= x(2-3x) = 0$$

1. $x = 0$

$$2 - 3x = 0$$
$$x = \frac{2}{3}$$

$$f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3$$
$$= \frac{4}{9} - \frac{8}{27}$$
$$= \frac{12}{27} - \frac{8}{27} = \frac{4}{27}$$

- Example:

$$f(x) = \sin x$$

$n\pi + \frac{\pi}{2}$

- Maxima at

$$x = 2n\pi + \frac{\pi}{2} \qquad n \text{ integer,}$$

- Minima at

$$x = 2n\pi + \frac{3\pi}{2} \qquad n \text{ integer.}$$

We have $f'(x) = \cos x = 0$ at those points.

sin

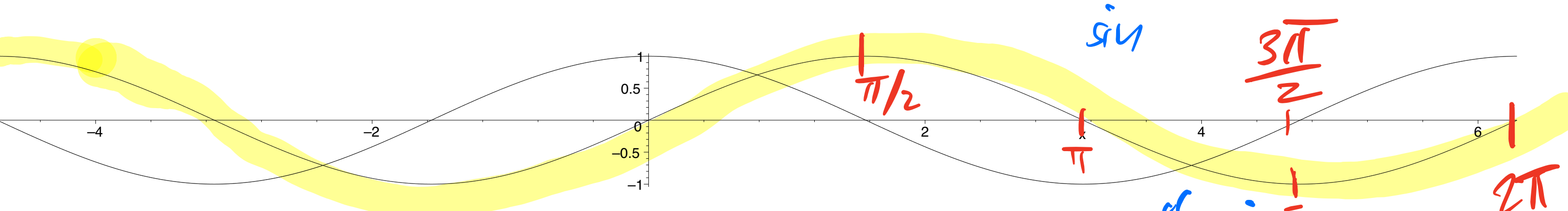
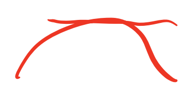


Figure 2. Sine and Cosine. $\cos x = \frac{d}{dx} \sin x$

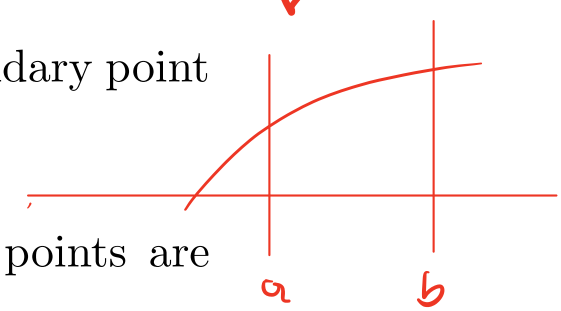
Vocabulary

- Suppose $y = f(x)$, f a given function.
- f has a **maximum value** $f(c)$ at $x = c$ if $f(c) \geq f(x)$ for all x in the domain of f.
- f has a **minimum value** $f(c)$ at $x = c$ if $f(c) \leq f(x)$ for all x in the domain of f.
- f has an **extreme value** $f(c)$ at $x = c$ if it has a minimum or maximum value at $x = c$.
- We need to refine this terminology later.
- So at what kind of points c can we get extreme values?
- It turns out that there are just three kinds:
- c is a **stationary point** if $f'(c) = 0$.



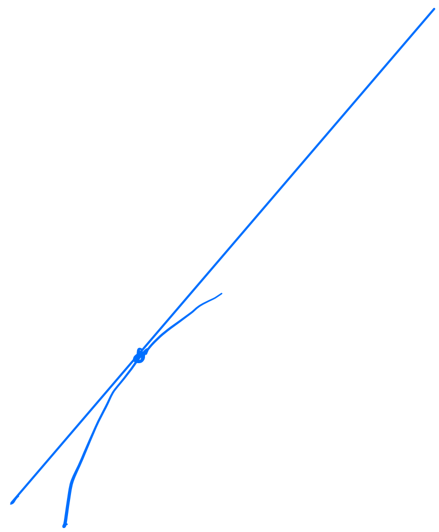
- c is a **singular point** if $f'(c)$ does not exist.

- c is a **boundary point** if is a boundary point of the domain of f.



- Collectively, these three kinds of points are called **critical points**.
- Extreme values can occur only at critical points!
- The word **point** refers to a value of the independent variable, not to a point in the plane.

- why only those three kinds of points?



- Example:

$$f(x) = -2x^3 + 3x^2 \qquad -\frac{1}{2} \le x \le 2$$

$\left[-\frac{1}{2}, 2\right]$

no singular pts

	CP	f(CP)
bdry pts	$-\frac{1}{2}$	1
	2	$-2\cdot 2^3+3\cdot 2^2 = -4$
stat pts	0	0
	1	1

stat. pts:

$f'(x) = -6x^2 + 6x$

$= -6x(x-1) = 0$

$-2\cdot\left(-\frac{1}{2}\right)^3 + 3\cdot\left(\frac{1}{2}\right)^2$

$= \frac{+2}{8} + \frac{3}{4} = 1$

- The calculations are consistent with the graph:

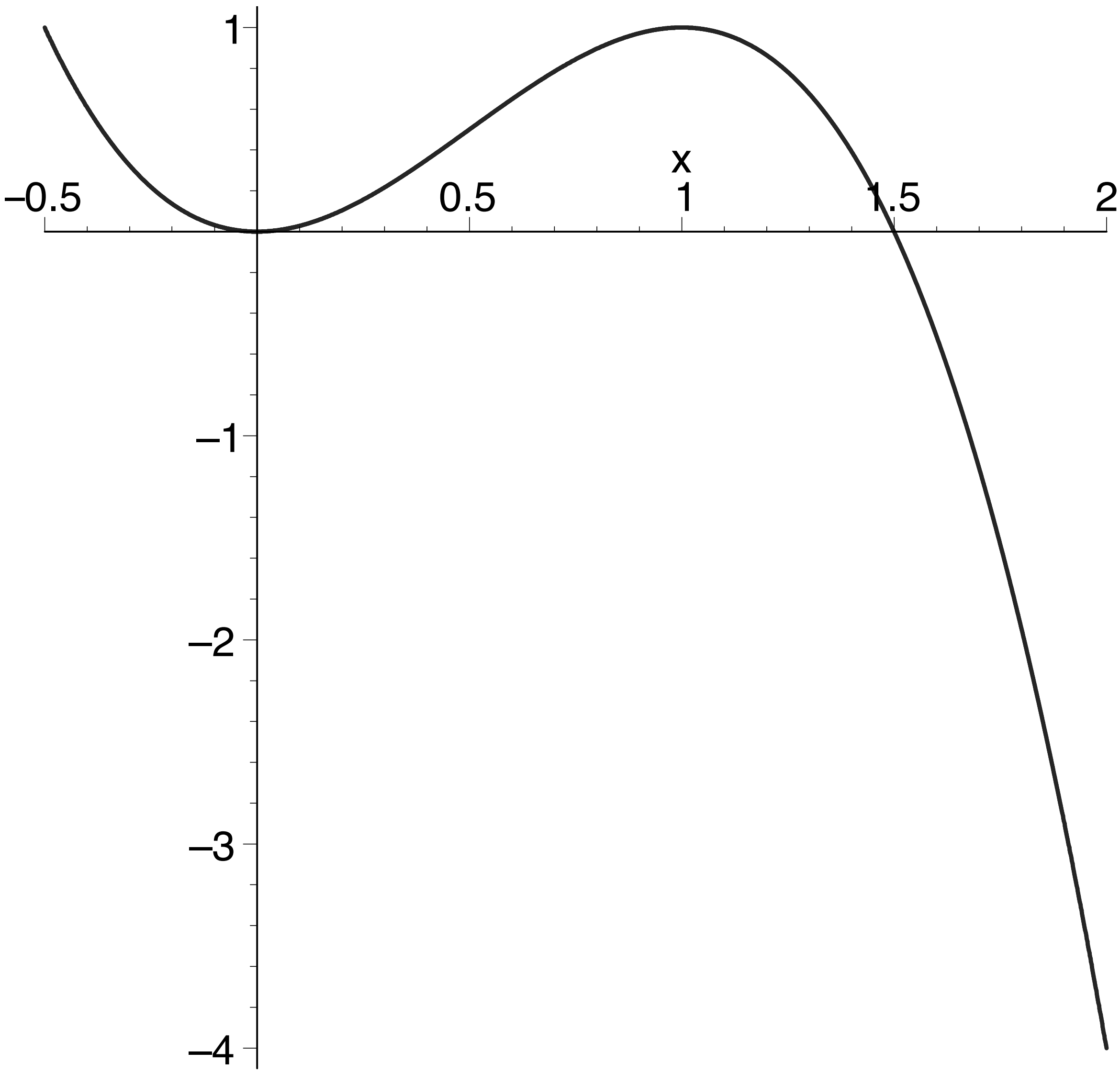


Figure 3. Graph of $f(x) = -2x^3 + 3x^2$, $-\frac{1}{2} \leq x \leq 2$.

- Example 2: $f(x) = x^3$, $-2 \le x \le 2$.

$f'(x) = 3x^2$

$f'(0) = 0$

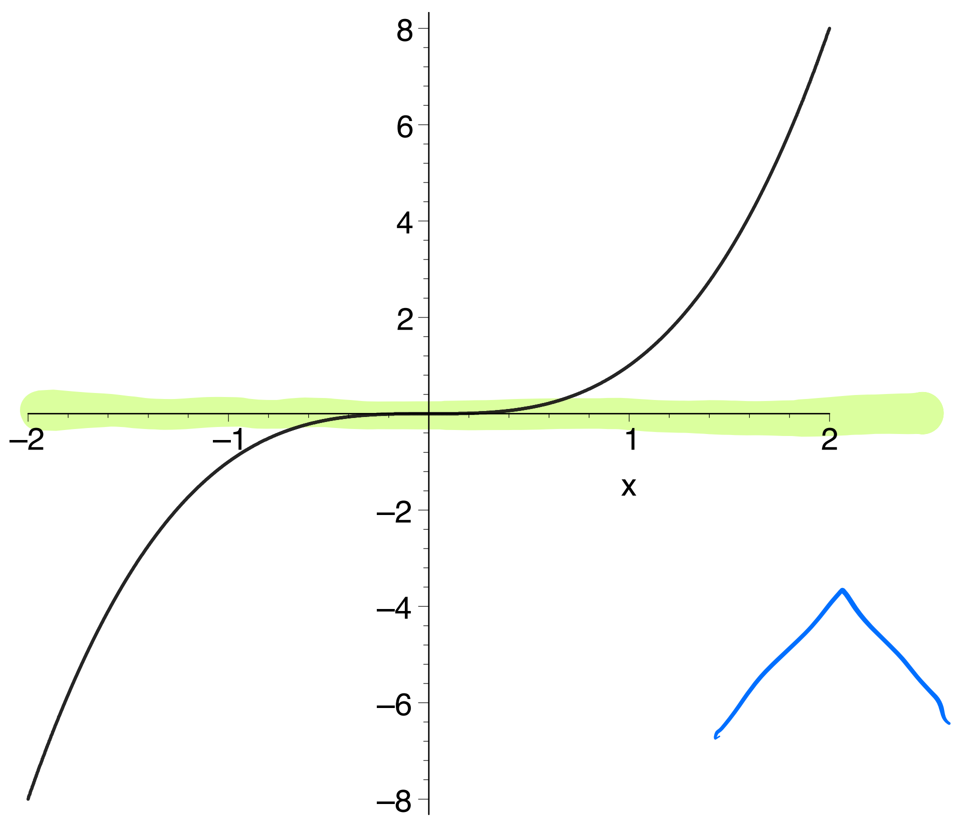


Figure 4. Graph of $f(x) = x^3$.

General Procedure

- So, to find extreme values of f, we proceed as follows:

1. Find all critical points (stationary, singular, or boundary) of f.

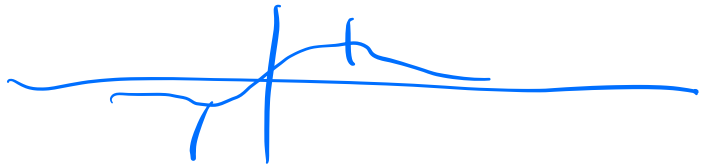
2. Evaluate f at those critical points.

3. Find the min and the max.

- Often there are no singular points, and in word problems what happens at the boundary points is often obvious.

- Example: Find the extreme values of

$$f(x) = \frac{x}{1+x^2} \qquad f(1) = \frac{1}{2}$$



$$f'(x) = \frac{1+x^2 - x \cdot 2x}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)} = 0$$

$$x = \pm 1$$

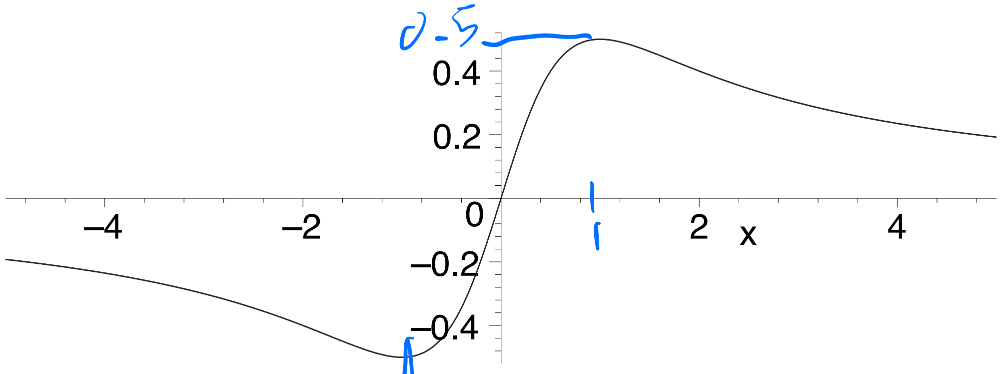
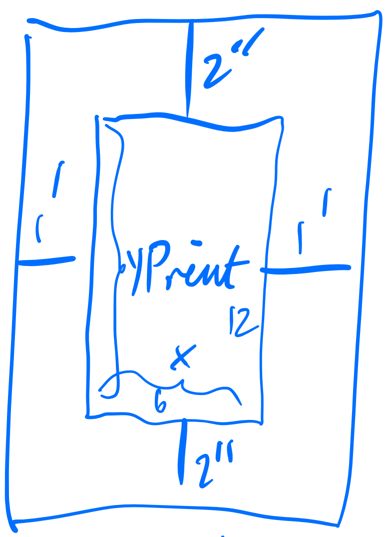


Figure 5. Graph of $f(x) = \frac{x}{1+x^2}$.

- A word problem: You design the layout of a (large, fancy, coffee-table) book. Each page should have an area of 72 square inches for text and pictures, 1 inch wide margins on the left and right, and 2 inch margins on top and bottom. What are the dimensions of each page that minimize the overall area (and cost) of each page?
- First think of expectations. Is the page going to be square, landscape (wider than high), or portrait (higher than wide)?
- How much higher? Twice as high, more than twice, less than twice?



$H = 16$ $\quad W = 12$

$A = WH = \text{min!}$

$xy = 72$

$y = \frac{72}{x}$

$$A = WH = (x+2)(y+4)$$

$$= (x+2)\left(\frac{72}{x} + 4\right)$$

$$= 72 + \frac{144}{x} + 4x + 8$$

$$= 80 + 4x + \frac{144}{x} = f(x)$$

$$f'(x) = 4 - \frac{144}{x^2} = 0$$

$$4 = \frac{144}{x^2}$$

$$\frac{x^2}{144} = \frac{1}{4}$$

$$x^2 = 144 \cdot \frac{1}{4} = 36$$

$$x^2 = 36$$

$x = 6$ $\qquad W = 6 + 2 = 8$

$y = \frac{72}{6} = 12$ $\qquad H = 12 + 4$
$= 16$