Math 1210-23

- No class or office hours on Monday!
- Study Session today, after class, right here!

Notes of 2/16/24

2.9 Differentials and Approximation

- An application of derivatives
- Recall that the Greek capital letter Δ (Delta) is often used to denote differences:

$$\Delta f = f(x_0 + \Delta x) - f(x_0)$$

- Recall our definition:

$$f'(x_0) = \lim_{\Delta x \longrightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

- rise over run, slope of secant approaches slope of tangent, average velocity approaches instantaneous velocity.
- We can turn things around, not take the limit, and think of the slope of the tangent (i.e., the derivative) as an **approximation** of the slope of the secant

$$f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

for some specific $\Delta x \neq 0$.

- This can be rewritten as

$$\boxed{\Delta f = f(x_0 + \Delta x) - f(x_0) \approx f'(x_0)\Delta x.}$$

- This can be considered the key property of the derivative:

The change in the function value equals approximately the change in the independent variable multiplied with the derivative.

- The approximation is the better the smaller the change in the independent variable.

Put differently: **To approximate the change in the output multiply the change in the input with the derivative!**

This makes the chain rule more plausible. When you compose two functions then each multiplies a change in input by its derivative, and so the composition multiplies with the product of the derivatives. Thus the derivative of the composition equals the product of the derivatives.

- Example: Suppose you need a good approximation of $\sqrt{4.6}$ and your calculator is broken.
- We can easily compute $\sqrt{4} = 2$.
- Let

$$f(x) = \sqrt{x} \qquad \Longrightarrow \qquad f'(x) = \frac{1}{2\sqrt{x}}.$$

- Then

$$\begin{aligned} f(4.6) - f(4.0) &= \sqrt{4.6} - \sqrt{4.0} \\ &\approx f'(4) \times (4.6 - 4.0) \\ &= \frac{4.6 - 4.0}{2\sqrt{4.0}} \\ &= 0.15 \end{aligned}$$

  and hence

$$\sqrt{4.6} = \sqrt{4} + (\sqrt{4.6} - \sqrt{4}) \approx 2 + 0.15 = 2.15.$$

- The difference between the true function value and its approximation is called the **error**.
- In this case $\sqrt{4.6} = 2.1448\ldots$ and the error is

$$\sqrt{4.6} - 4.15 = -0.0052\ldots.$$

- This works the better the smaller the difference in inputs. In this case this means the number whose square root we want to approximately is closer to 4.

- Consider this Table:

Δx	$x_0 + \Delta x$	$f(x_0 + \Delta x)$	$f(x_0) + f'(x_0)\Delta x$	
Δx	$4 + \Delta x$	$\sqrt{4 + \Delta x}$	$2 + \frac{\Delta x}{\sqrt{4}}$	error
0.6	4.6	2.144761059	2.150000000	-0.005238
0.3	4.3	2.073644135	2.075000000	-0.001356
0.15	4.15	2.037154879	2.037500000	-0.000345
0.075	4.075	2.018662924	2.018750000	-0.000087

Note that as we go from one line to the next the value of Δx is halved. On the other hand, the **error** is reduced by a factor about 4.

- This means the error goes to zero faster than linearly.

- In a profound way this behavior can be used to define the derivative, but that subject is beyond our scope.

Linear Approximation

- We saw that in general

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

- The right hand side of this approximation is a linear function of x. (Its graph of course is the tangent line at $\big(x_0, f(x_0)\big)$. It is called the **linear approximation** of f at (or about) x_0.
- Some examples for linear approximations:

$$\begin{array}{ccccccc} f(x) & = & \sqrt{x} & \approx & 2 + \frac{x-4}{2\sqrt{4}} & = & 2 + \frac{x-4}{4} & (x_0 = 4) \\ f(x) & = & \sin x & \approx & \sin(0) + (x-0)\cos 0 & = & x & (x_0 = 0) \\ f(x) & = & x^2 & \approx & 1^2 + 2 \times 1 \times (x-1) & = & 1 + 2(x-1) & (x_0 = 1) \end{array}$$

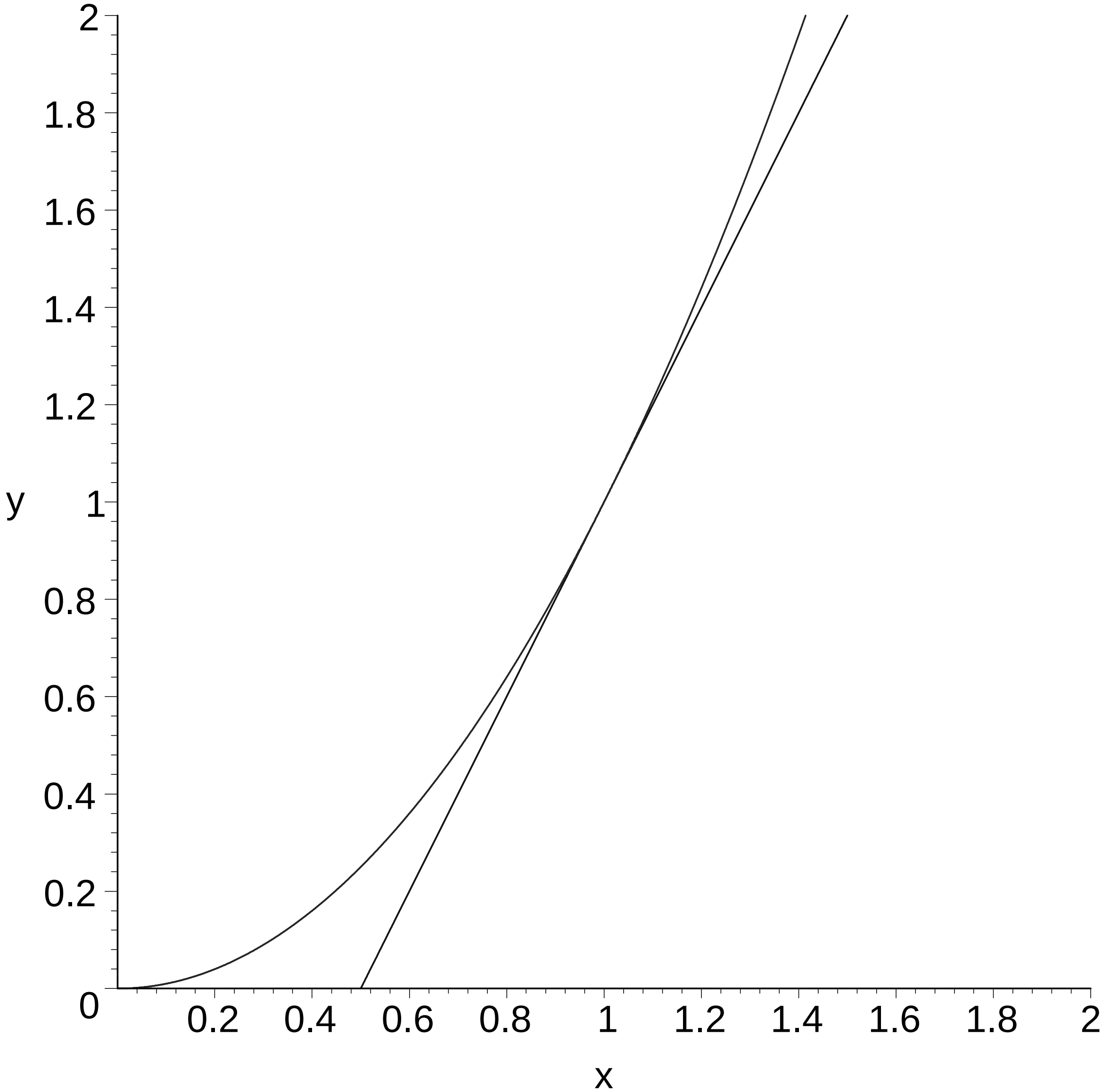

Figure 1. Linear Approximation of $f(x) = x^2$.

- Another Example: You manufacture cubes. Their nominal length s (and width and height) are 12 inches. How will an error of Δs effect the volume (and weight and material cost) of the cube?

- Here is a similar Table as in the first example:

Δs	$s_0 + \Delta s$	$f(s_0 + \Delta s)$	$f(s_0) + f'(s_0)\Delta s$	
Δs	$12 + \Delta s$	$(12 + \Delta s)^3$	$12^3 + 3 \times 144 \times \Delta s$	error
1	13	2197	2160	37
0.1	12.1	1771.561	1771.2	0.361000
0.01	12.01	1732.323601	1732.32	0.003601
0.001	12.001	1728.432036	1728.432	0.000036

- In this case we reduce the change by a factor 10 each time, and the error is reduced by a factor $100 = 10^2$.

Notation

- The notation and terminology is confusing. Don't loose track of the key idea: change in output equals approximately the change in input multiplied with the derivative.
- We consider the equation $y = f(x)$. x is the independent variable, y the dependent variable.
- Δx is an arbitrary increment in the independent variable.
- $\mathrm{d}x = \Delta x$ is called the **differential** of the independent variable x.
- $\Delta y = \Delta f = f(x + \Delta x) - f(x)$ is the actual change in the dependent variable (or the function, or the output).
- $\mathrm{d}y$ is the **differential** of the dependent variable y, defined by

  $$\mathrm{d}y = f'(x)\mathrm{d}x \tag{1}$$

  In this context you can think of both $\mathrm{d}y$ and $\mathrm{d}x$ as ordinary variables. The equation (1) is extremely suggestive. It can be rewritten as

  $$f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} \tag{2}$$

- Previously we thought of (2) as a short hand notation for derivatives. Now we are thinking of the differential $\mathrm{d}x$ and $\mathrm{d}y$ as variables.

- However, with this new notation we have the key fact

$$\Delta y \approx \mathrm{d}y = f'(x)\mathrm{d}x$$

- Again, this captures the key idea: change in output equals approximately the change in input multiplied with the derivative.

- Another Example. Let's consider a cylinder. How does a relative error in radius ($\frac{\Delta r}{r}$) or height ($\frac{\Delta h}{h}$), affect the relative error in the volume ($\frac{\Delta V}{V}$)?
- The relative error might be measured in percent.

- Casually speaking, when making cylinders, it's more important (twice as important!) to get the radius right, than to get the height right.

Aside: More Confusion

- Using the Leibniz notation

$$f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}$$

  and at the same time thinking of the differentials as variables can lead to confusing notation.

- For example, recall the product rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(uv) &= uv' + vu' \\ &= u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}\end{aligned} \tag{3}$$

- "Multiplying with $\mathrm{d}x$" turns this into

$$\mathrm{d}(uv) = u\mathrm{d}v + v\mathrm{d}u \tag{4}$$

- The textbook does use this notation in some places. See the Table on page 143 (and the box in the margin next to that Table). I think of that notation as extremely confusing, and avoid it when possible. You can go from (4) to (3) by "dividing by $\mathrm{d}x$".