Math 1210-4

Notes of 09/27/17

• Recall our differentiation rules:

\[(kf)' = kf'\] \hspace{1cm} \text{Constant Multiple Rule}

\[(f + g)' = f' + g'\] \hspace{1cm} \text{Sum Rule}

\[(x^n)' = nx^{n-1}\] \hspace{1cm} \text{Power Rule (n integer)}

\[(fg)' = f'g + fg'\] \hspace{1cm} \text{Product Rule}

\[
\left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2} \hspace{1cm} \text{Quotient Rule}
\]

\[
\frac{d}{dx} \sin x = \cos x \hspace{1cm} \text{Sine Rule}
\]

\[
\frac{d}{dx} \cos x = -\sin x \hspace{1cm} \text{Cosine Rule}
\]

\[
\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \hspace{1cm} \text{Chain Rule}
\]

\[
\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}} \hspace{1cm} \text{Square Root Rule}
\]
2.7 Implicit Differentiation

- As usual, this is only a small extension of what we did previously. But it may still surprise you.

- We’ve been talking about how to differentiate, using our differentiation rules.

- That’s what we’ll continue to do today.

- But we won’t necessarily assume that our function is defined explicitly.

- Example:

\[
\frac{d}{dx} \sqrt{x} = \frac{1}{2 \sqrt{x}} = f'(x)
\]

\[
f(x) = \sqrt{x}
\]

\[
y = \sqrt{x}
\]

\[
y^2 = x
\]

\[
y = y(x)
\]

\[
2yy' = 1
\]

\[
y' = \frac{1}{2y} = \frac{1}{2\sqrt{x}}
\]
Example: Do

\[ f(x) = \sqrt{1 - x^2} = y \]

\[ 1 - x^2 = y^2 \]

explicitly and implicitly

\[ x^2 + y^2 = 1 \]

\[ f(x) = (1 - x^2)^{\frac{1}{2}} \]

\[ f'(x) = \frac{1}{2} (1-x^2)^{-\frac{3}{2}} (-2x) \]

\[ = -x (1-x^2)^{-\frac{3}{2}} \]

\[ = \frac{-x}{(1-x^2)^{\frac{3}{2}}} \]

\[ = \frac{-x}{\sqrt{1-x^2}} \]

\[ x^2 + y^2 = 1 \Rightarrow 2x + 2y \cdot y' = 0 \Rightarrow y' = -\frac{x}{y} \]
• Another example:

\[ y = y(x), \quad y^3 + 7y = x^3 \]

\[ x = 2 \quad y = 1 \quad \Rightarrow \quad 1^3 + 7 = 2^3 = 8 \]

• We think of \( y \) as a function of \( x \)

• It’s easy to check that the point \((2, 1)\) is on the graph.

• What is \( y' (2) \)?

\[ 3y^2y' + 7y' = 3x^2 \]

\[ y' (3y^2 + 7) = 3x^2 \]

\[ y' = \frac{3x^2}{3y^2 + 7} \]

\[ x = 2 \quad y = 1 \quad \Rightarrow \quad y' = \frac{12}{10} \]

\((2, 1)\)
• We computed the derivative (in terms of $x$ and $y$) at $x = 2$ without actually having an expression for $y$ in terms of $x$.

• Pretty Cool!

• It’s crucial to **differentiate first and evaluate second**.

• If we were to evaluate at $x = 2$ and $y = 1$ in

\[ y^3 + 7y = x^3 \]

we’d get

\[ 8 = 8 \]

\[ \text{circled } 8 = 8 \]

• Differentiating in that equation gives

\[ 0 = 0 \]

which is true but does not tell us about $y'(2)$. 
- Profound application:

\[ \frac{d}{dx} x^r = r x^{r-1} \]

for all rational numbers \( r \).

- At this stage, technically we know the power rule only if \( r \) is an integer.

\[
\begin{align*}
\rho &= \frac{p}{q} \\
\rho, q \text{ integers} \\
q \neq 0 \\
y &= x^{\rho/q} \\
y^q &= x^\rho \\
y^{q-1} &= p x^{p-1} \\
\frac{d}{dx} y^{q-1} &= \frac{p}{q} x^{p-1} \\
\frac{d}{dx} y &= \frac{p}{q} x^{p-1} \\
\frac{d}{dx} \left( x^{\rho/q} \right)^{q-1} &= \frac{p}{q} x^{\rho/q} \\
\therefore \quad y &= x^{\rho/q}
\end{align*}
\]
\[
\frac{p \times p^{-1}}{\frac{p}{q} (q - 1)} = \frac{p}{q} \times \frac{p}{q} (q - 1) = \frac{p}{q} (p - 1) = \frac{p}{q} - 1
\]
• Let’s do another example where we can get the same answer both ways and practice differen-
tiation as well.

• Example 1, simplified:

\[ x^2 y - y = x^3 - 1 \]

\[
\gamma(x-1)(x+1) = \gamma(x^2-1) = x^3 - 1 = (x-1)(x^2 + x + 1)
\]

\[
\gamma (x + 1) = x^2 + x + 1
\]

\[
\gamma = \frac{x^2 + x + 1}{x + 1}
\]

**Explicit:** \[ \gamma' = \frac{(2x+1)(x+1) - (x^2 + x + 1)}{(x+1)^2} \]

\[ = \frac{2x^2 + 3x + 1 - x^2 - x - 1}{(x + 1)^2} \]

\[ = \frac{x^2 + 2x}{(x + 1)^2} \]

\[ \gamma' = \frac{x^2 + 2x}{(x + 1)^2} \]
\[ x^2 y - y = x^3 - 1 \]
\[ 2x^2 y + x^2 y' - y' = 3x^2 \]
\[ y'(x^2 - 1) = 3x^2 - 2xy \]
\[ y' = \frac{3x^2 - 2xy}{x^2 - 1} \]
\[ y = \frac{x^2 + x + 1}{x + 1} \]
\[ y' = \frac{3x^2 - 2x \cdot \frac{x^2 + x + 1}{x+1}}{x^2 - 1} \]
\[ = \frac{3x^2 (x+1) - 2x(x^2+x+1)}{(x^2-1)(x+1)} \]
\[ = \frac{3x^3 + 3x^2 - 2x^3 - 2x^2 - 2x}{(x-1)(x+1)(x+1)} \]
\[ = \frac{x^3 + x^2 - 2x}{(x-1)(x+1)^2} \]
\[ = \frac{x^2 + 2x}{(x+1)^2} \]
Another Example, if time permits. Problem 47. The graph of

\[ x^2 - xy + y^2 = 16 \]

is a tilted ellipse. Find the equations of the tangents at the \( x \) intercepts.

\[ y=0 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4 \]

\[ 2x - y - xy' + 2yy' = 0 \]

\[ y' (2y - x) = y - 2x \]

\[ y' = \frac{y - 2x}{2y - x} \]

\[ x = \pm 4 \]

\[ y = 0 \]

\[ y' = \frac{-2x}{-x} = 2 \]