

# Math 1210-23

## Notes of 2/13/24

- Anna Alfeld: The hardest part of Calculus is

## 2.8 Related Rates

- “Rate”: derivative with respect to time.
- “related”: just that, several time derivatives occur in the same problem, and they are connected in some way.
- time usually does not occur explicitly.



Suggestion: use variables and substitute constants only at the end.

- I got several questions after class yesterday about the first example. As a review, here is that example again, in print, and in slow motion:
- You blow air into a spherical balloon at the rate of 1 cubic foot per minute. How fast is the radius  $r$  of the balloon increasing when the radius equals 1 foot? 2 feet?
- quick review: the Volume  $V$  and surface area  $A$  of a sphere of radius  $r$  are given by



$$V = \frac{4\pi}{3}r^3 \quad \text{and} \quad A = 4\pi r^2.$$

- in our problem,  $r$ ,  $A$ , and  $V$  are functions of time. We use a prime to denote differentiation with respect to time, but time will nowhere occur explicitly. We don't even need a variable for it!

- However, we know that

$$V' = 1 \quad \text{cubic foot per minute}$$

since the problem says that the volume increases by 1 cubic foot per minute.

- We start with the volume equation:

$$V = \frac{4\pi}{3}r^3$$

- Differentiating in that equation gives

$$V' = \frac{4\pi}{3}(3r^2r') = 4\pi r^2r'$$

- Solving for  $r'$  gives

$$r' = \frac{V'}{4\pi r^2} \quad \text{feet/minute.}$$

- Substituting  $V' = 1$  and  $r = 1$  or  $2$  feet gives:

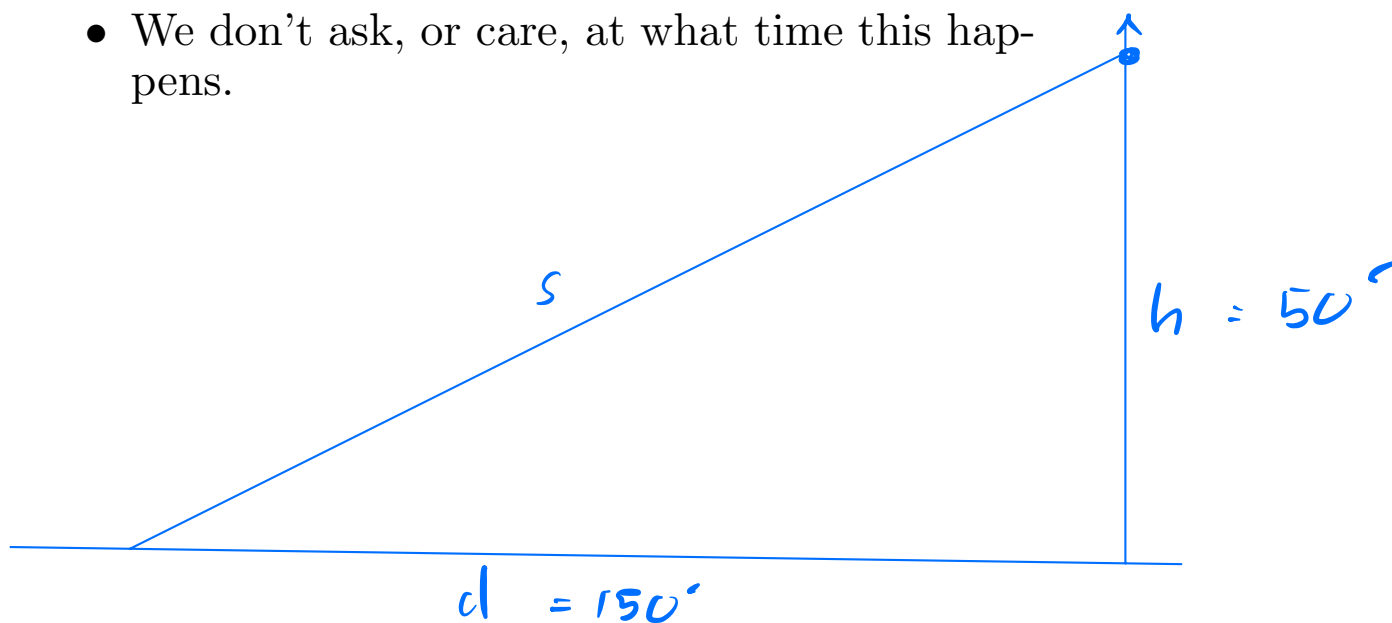
$$r = 1 \quad \text{foot} \quad \implies \quad r' = \frac{1}{4\pi 1^2} \quad \text{feet/minute} \approx 0.95 \quad \text{inches/minute,}$$

and

$$r = 2 \quad \text{feet} \quad \implies \quad r' = \frac{1}{4\pi 2^2} \quad \text{feet/minute} \approx 0.24 \quad \text{inches/minute.}$$

- The rate with which the radius is increasing is inversely proportional to the square of the radius.

- Example 1, textbook: A small balloon is released at a point 150 feet from an observer on level ground. If the balloon goes straight up at 8 feet per second, how fast is the distance from the observer increasing when the balloon is 50 feet high?
- Subtlety: Assume the observer is at ground level, or, alternatively, height is measured from the height of the observer.
- Let  $s$  be the distance from the observer, and  $h$  be height.
- Both depend on time.
- Typical: several quantities ( $s$  and  $h$ ) depend on time. They are related. Hence the phrase “related rates”.
- Here we ask what is  $s'$  when  $h = 50$ .
- We don't ask, or care, at what time this happens.



$$d^2 + h^2 = s^2 \quad d \text{ constant}$$

$$2hh' = 2ss'$$

$$hh' = ss'$$

$$s' = \frac{h}{s} h' = \frac{hh'}{s}$$

$$s' = \frac{h}{\sqrt{d^2 + h^2}} h'$$

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$$s' = \frac{50}{\sqrt{150^2 + 50^2}} \text{ } \text{\textcircled{8}}$$

$$= \frac{50}{50 \sqrt{3^2 + 1^2}} \text{ } \text{\textcircled{8}} = \frac{8}{\sqrt{10}}$$

$$\approx 2.5 \text{ f/m}$$



As an exercise you may want to do this the old clumsy way where you express everything in terms of  $t$ , solve for the time when you are at the height  $h$ , and substitute that time value into the formula for  $s'$ .



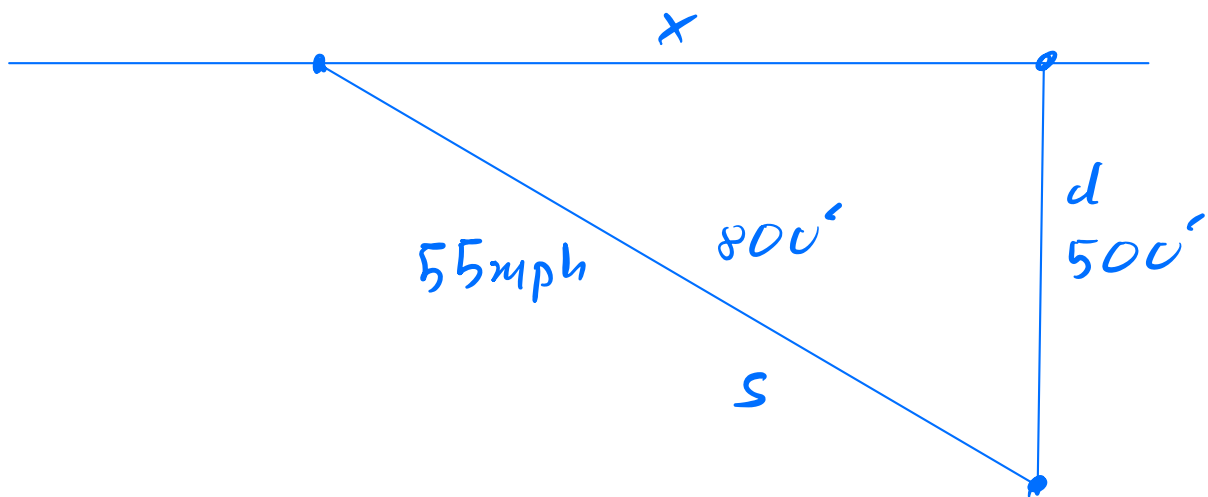
- Note: we must differentiate before we evaluate. If we evaluate first we get

$$s = \sqrt{150^2 + 50^2} = \sqrt{25,000}$$
$$s' = 0$$

- This is sweet and short, but false.

- Here is a modification of this problem.
- A radar gun measures the velocity at which a car is moving towards you or away from you. It does not measure the speed or velocity as such.
- You are 500 feet away from a highway. You measure the speed of a car moving along the highway towards you at 55mph when the car is 800 feet away from you. The speed limit is 65mph. Is he speeding?

65mph



$$x' = ?$$

$$x^2 + d^2 = s^2$$

$$x^2 = s^2 - d^2$$

$d$  const.

$$x = \sqrt{s^2 - d^2}$$

$$2xx' = 2ss'$$

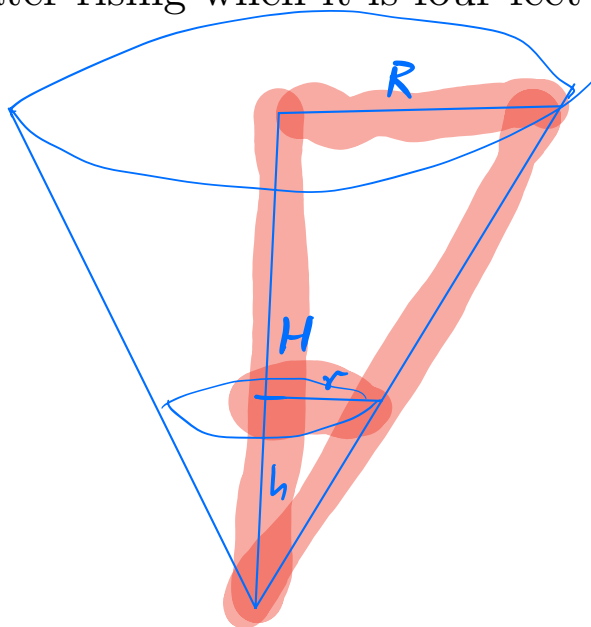
$$x' = \frac{s}{x} s'$$

$$= \frac{s}{\sqrt{s^2 - u^2}} s'$$

$$x' = \frac{800}{\sqrt{800^2 - 500^2}} \quad 55$$

$$x' = \frac{8}{\sqrt{8^2 - 5^2}} \quad 55$$

- Example 2. Water is pouring into a tank in the shape of an inverted cone, at the rate of 8 cubic feet per minute. The tank is 12 feet high and its radius is 6 feet on top. How fast is the water rising when it is four feet deep?



$$\begin{aligned}
 H &= 12' \\
 R &= 6' \\
 h &= 4' \\
 h' &= ? \\
 V' &= 8 \text{ f}^3/\text{min}
 \end{aligned}$$

$$\begin{aligned}
 V &= \frac{\pi r^2 h}{3} \\
 V &= \frac{\pi R^2 h^2 h}{H^2 3} & \frac{r}{h} &= \frac{R}{H} h' \\
 & & r &= \frac{R}{H} h
 \end{aligned}$$

$$= \frac{\pi R^2}{3 H^2} h^3$$



$$V' = \frac{\pi R^2}{3H^2} \cdot 3h^2 h'$$

$$= \frac{\pi R^2}{H^2} h^2 h'$$

$$h' = \frac{H^2}{\pi R^2 h^2} V'$$

$$h' = \frac{12^2}{\pi 6^2 4^2} \cdot 8 = \frac{4}{2\pi}$$

$$\frac{12^2}{6^2} = 4$$

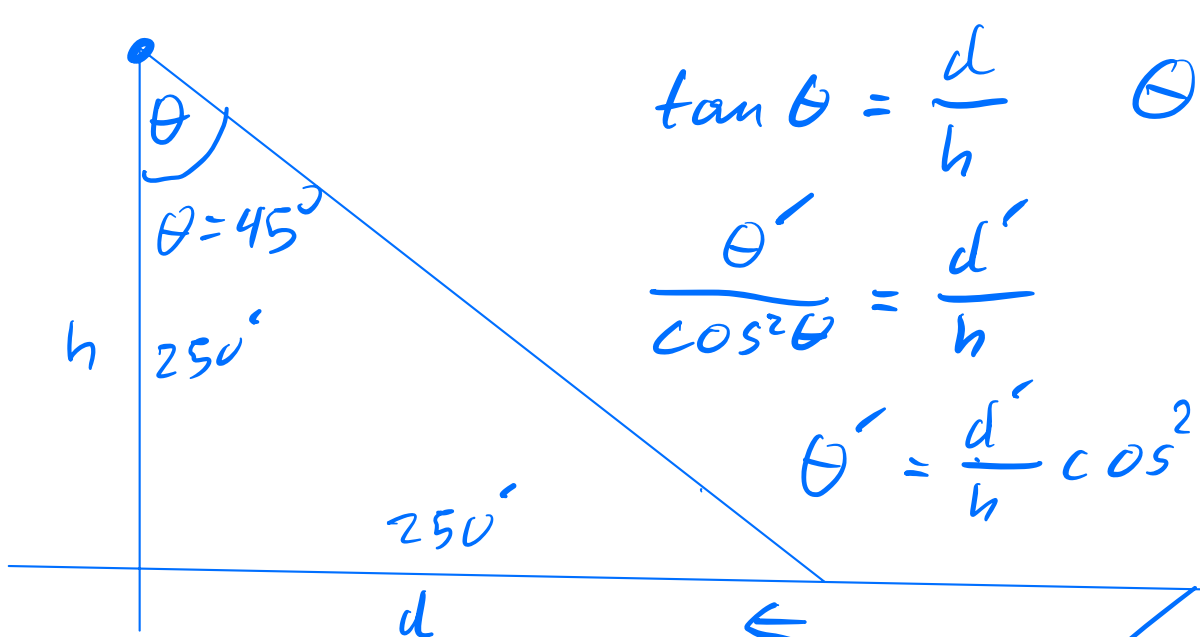
$$\frac{8}{4^2} = \frac{1}{2}$$

$$\frac{8}{4^2} = \frac{8}{16} = \frac{1}{2}$$

$$= \frac{2}{\pi} \text{ ft/m}$$

- Example 4, modified. A woman standing on top of a cliff is watching a motorboat as the boat approaches the shoreline directly below her. Let  $\theta$  be the angle that her line of sight makes with the vertical. If the boat is approaching at  $v$  feet per second and the height of the cliff is  $h$  feet, how fast is  $\theta$  changing when the boat is  $d$  feet from the shoreline? Assume the boat is moving perpendicularly to the shoreline, and the earth is flat. (Query: What's the significance of these assumptions?) To get the special case given in the original example consider the special case that  $v = 20$  feet per second, and  $h = d = 250$  feet.
- Query: Do you expect  $\theta'$  to be positive or negative? What about the sign of  $\theta''$ ?

$$d' = v = -20 \text{ ft/s}$$



$$\tan \theta = \frac{d}{h} \quad \ominus'$$

$$\frac{\theta'}{\cos^2 \theta} = \frac{d'}{h}$$

$$\theta' = \frac{d'}{h} \cos^2 \theta$$



## Summary

1. Assign variables to quantities that depend on time (and by all means feel free to use variables instead of specific constants, to get a more general solution, and to facilitate dimension checking).
2. Write one or more equations relating the variables, and encapsulating what is known about the problem.
3. Differentiate with respect to time in those equations, keeping clearly in mind which variables do depend on time, and which do not.
4. Solve for required variables or their derivatives.
5. Substitute specific values for the variables.
  - You must have variables when you differentiate. If you substitute constants for variables before differentiating you get  $0=0$ .
  - Because of the difficulty of the subject we will have another session on Related Rates tomorrow.