Math 1210-23, Spring 2024

Notes of 2/12/24

- Recall our differentiation rules:

$(kf)'$	$=$	kf'	**Constant Multiple Rule**
$(f+g)'$	$=$	$f'+g'$	**Sum Rule**
$(x^r)'$	$=$	rx^{r-1}	**Power Rule (r rational)**
$(fg)'$	$=$	$f'g+fg'$	**Product Rule**
$\left(\frac{f}{g}\right)'$	$=$	$\frac{f'g-fg'}{g^2}$	**Quotient Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\sin x$	$=$	$\cos x$	**Sine Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\cos x$	$=$	$-\sin x$	**Cosine Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}f\big(g(x)\big)$	$=$	$f'\big(g(x)\big)g'(x)$	**Chain Rule**

More on Implicit Differentiation

- Section 2.7 continued
- Example 3, textbook. Find the equation of the tangent line to the curve

$$y^3 - xy^2 + \cos xy = 2 \tag{1}$$

at the point $(0, 1)$.

$y = y(x)$

$' = \frac{d}{dx}$

$3y^2 y' - (y^2 + x 2 y y')$

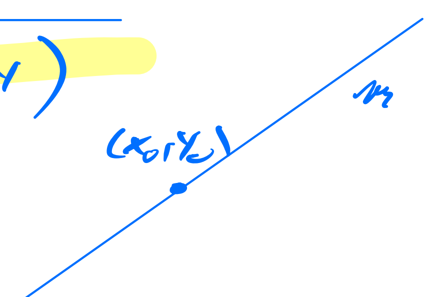
$3y^2 y' - y^2 - 2xyy' - \sin(xy)(y + xy') = 0$

$y'(3y^2 - 2xy - x\sin(xy)) = y^2 + \sin(xy)\, y$

$$y' = \frac{y^2 + \sin(xy)\, y}{3y^2 - 2xy - x\sin(xy)}$$

$x = 0 \quad y = 1$

$y' = \frac{1}{3}$



$\frac{y - y_0}{x - x_0} = m$

$\frac{1 - y}{0 - x} = \frac{1}{3}$

$1 - y = -\frac{1}{3}x \qquad y = 1 + \frac{1}{3}x$

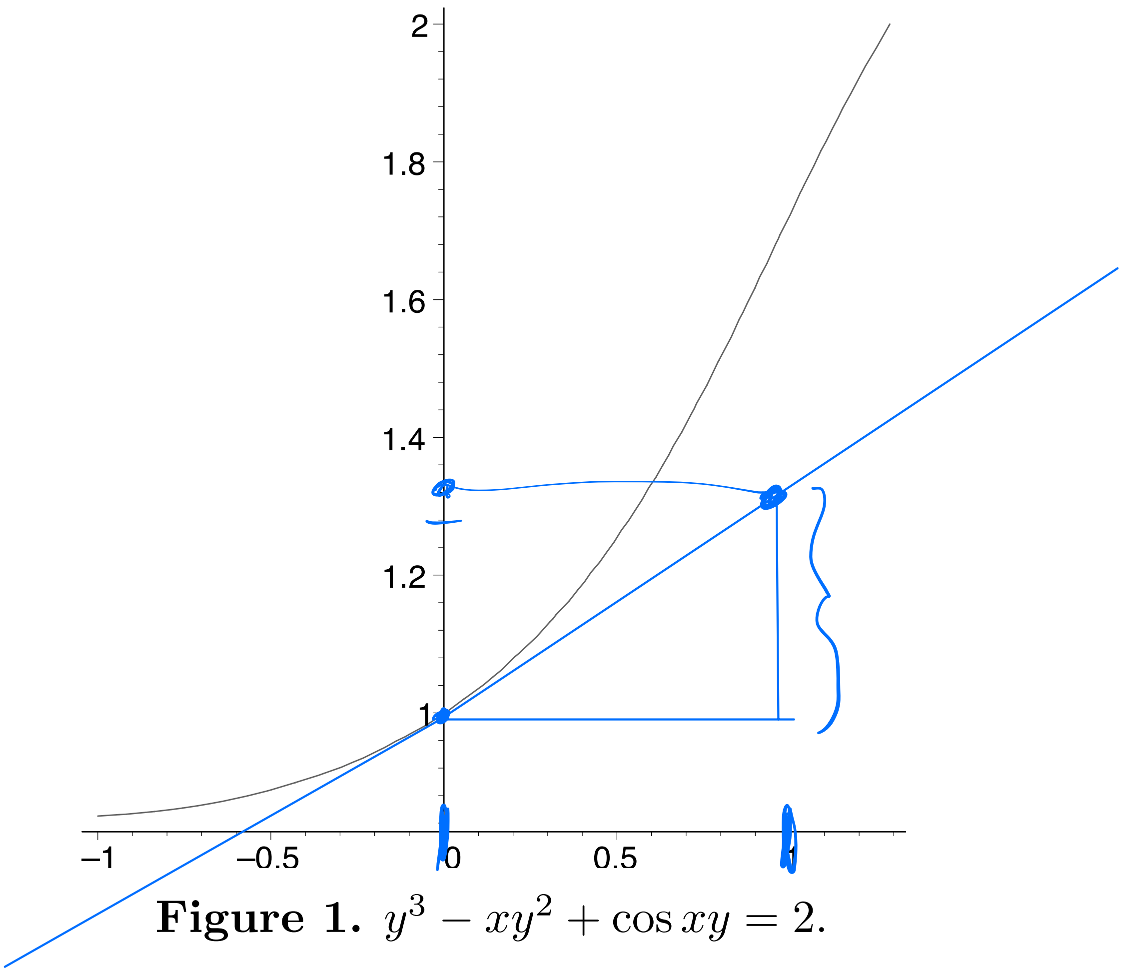


Figure 1. $y^3 - xy^2 + \cos xy = 2.$

- Figure 1 shows the graph of equation (1).

Naturally, implicit differentiation can be done more than once.

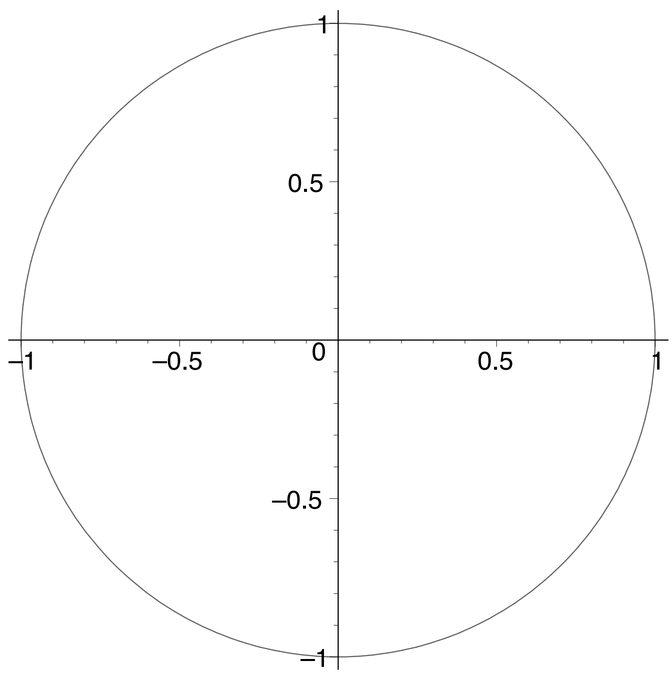


Figure 2. $x^2 + y^2 = 1$.

- Example: Suppose the function $y = f(x)$ is defined by the equation

$$x^2 + y^2 = 1. \tag{2}$$

  Compute y' and y'' and express it in several ways.

Of course, the graph of (2) is the unit circle, see Figure 2. It does not define a single function, it fails the vertical line test. We can do our stuff only locally, on part of the unit circle.

$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

$\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2}$

$$x^2+y^2=1$$

$$y^2=1-x^2$$

$$y=\sqrt{1-x^2}=(1-x^2)^{1/2}$$

$$y'=\frac{-2x}{2\sqrt{1-x^2}}=-\frac{x}{\sqrt{1-x^2}}$$

$$y''=-\frac{\sqrt{1-x^2}-x\cdot\frac{x}{\sqrt{1-x^2}}}{1-x^2}\cdot\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}$$

$$=-\frac{1-x^2+x^2}{(1-x^2)^{3/2}}=-\frac{1}{(1-x^2)^{3/2}}$$

$$(1-x^2)\sqrt{1-x^2}=(1-x^2)^1\cdot(1-x^2)^{1/2}$$

$$x^2+y^2=1$$

$$2x+2yy'=0$$

$$x+yy'=0$$

$$y'=-\frac{x}{y}$$

$$y'=-\frac{x}{y}$$

$$y'^2=x$$

$()'$

$$1+y'^2+yy''$$

$$yy''=-(1+y'^2)$$

$$y''=-\frac{1+y'^2}{y}$$

$$=-\frac{1+\frac{x^2}{y^2}}{y}=-\frac{y^2+x^2}{y^3}$$

$$=-\frac{1}{y^3}$$

- Example: Compute y' where $y = f(x)$ is defined by the equation

$$\sin(y) = x$$

and y is in the interval $\left[-\frac{\pi}{2}\frac{\pi}{2}\right]$.

$()'$ $\quad y' \cos y = 1$

$$y' = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1 - \sin^2 y}}$$

$$= \frac{1}{\sqrt{1 - x^2}}$$

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$\sin x = \sqrt{1 - \cos^2 x}$

$\cos x = \sqrt{1 - \sin^2 x}$

$$\frac{1}{\sqrt{1-x^2}}$$

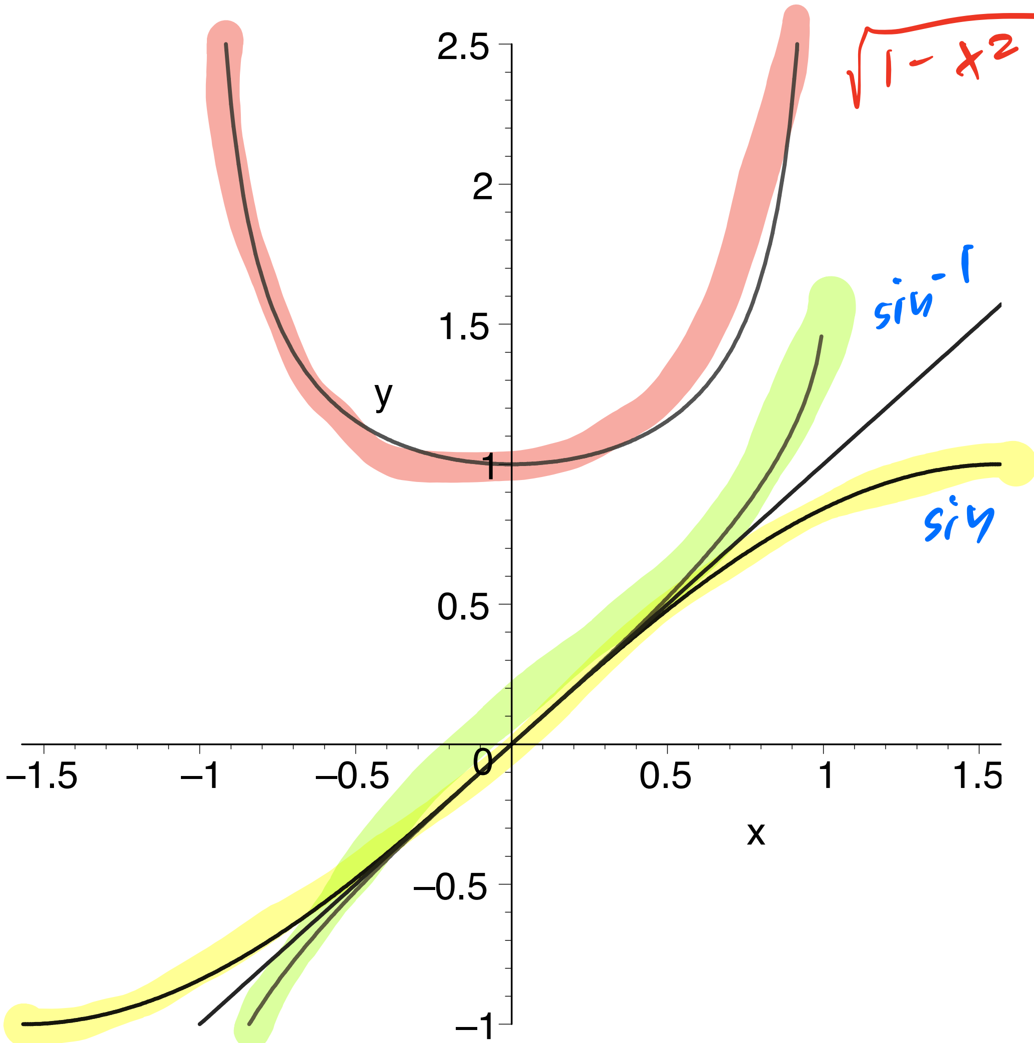
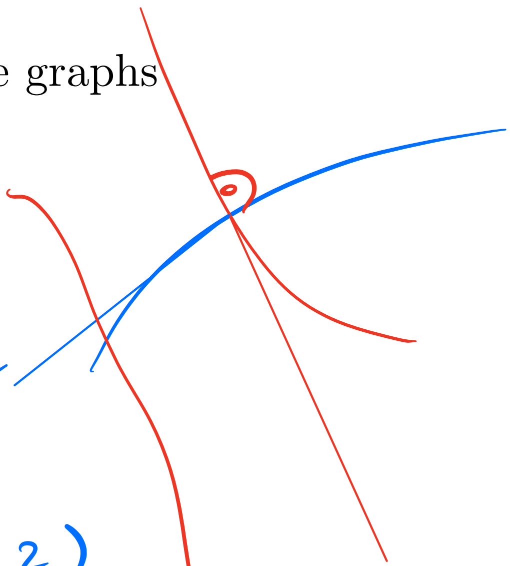


Figure 3. $\sin(y) = x$.

- Problem 43, page 134. Show that the graphs of

$$2x^2 + y^2 = 6 \quad \text{and} \quad y^2 = 4x$$

intersect at right angles.



$y^2 = 4$

$y = \pm 2$

$2x^2 + 4x = 6$

$x^2 + 2x = 3$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$\cancel{x = -3}$ or $x = 1$

$(1, 2)$

$(1, -2)$

$2yy' = 4$

$y' = \dfrac{4}{2y}$

$x = 1$, $y = 2$: $y' = +1$

$4x + 2yy' = 0$

$y' = -\dfrac{4x}{2y}$

$x = 1$, $y = 2$

$y' = -1$

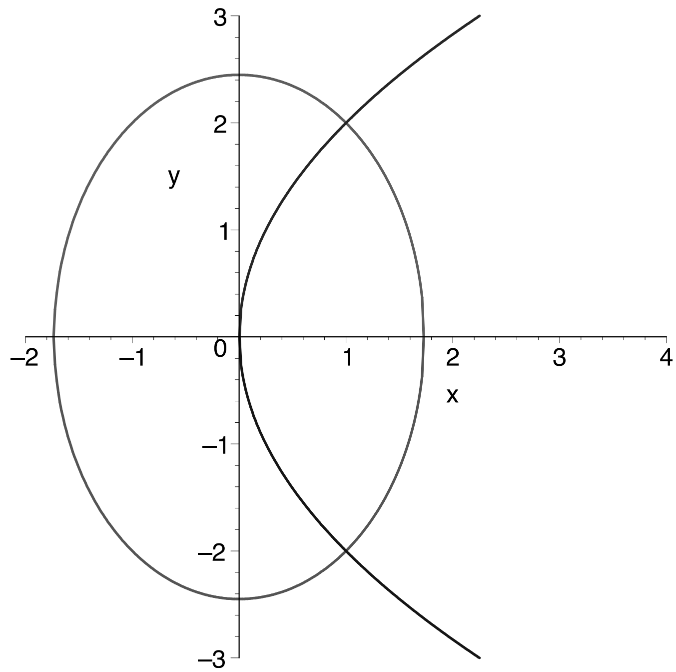
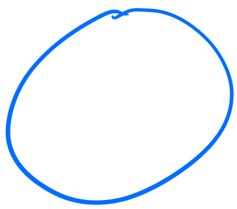


Figure 4. Intersection of $2x^2 + y^2 = 6$ and $y^2 = 4x$.

The intersections are shown in Figure 4.

2.8 Related Rates

- Example: You are blowing air into a balloon at the rate of one cubic foot per minute. How fast is the radius growing when the radius is one foot? Two feet?



$ft^3 \quad = V = \frac{4}{3}\pi r^3 \qquad r' = ? \qquad r = 1\,ft$

$ft^3/min \quad 1 = V' = 4\pi r^2 r' \qquad \frac{d}{dt} r^3 = 3r^2 r'$

$$r' = \frac{1}{4\pi r^2}$$

$r=1 \quad r' = \frac{1}{4\pi} \qquad \approx 1''/min$

$r=2 \quad r' = \frac{1}{4\pi 4} \qquad \approx 1/4''/min$