Math 1210-23

Notes of 2/9/24

- We are heading into some tricky territory and will spend two meetings on section 2.7 of our textbook.
- Recall our differentiation rules:

$(kf)' = kf'$		**Constant Multiple Rule**
$(f+g)' = f' + g'$		**Sum Rule**
$(x^n)' = nx^{n-1}$		**Power Rule (n integer)**
$(fg)' = f'g + fg'$		**Product Rule**
$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$		**Quotient Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$		**Sine Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$		**Cosine Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$		**Chain Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{1}{2\sqrt{x}}$		**Square Root Rule**

2.7 Implicit Differentiation

- As usual, this is only a small extension of what we did previously. But it may still surprise you.
- We've been talking about how to differentiate, using our differentiation rules.
- That's what we'll continue to do today.
- But we won't necessarily assume that our function is defined explicitly.
- Example:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$y = \sqrt{x}$

$y^2 = x \quad |()'$

$y = y(x)$

$' = \frac{d}{dx}$

$2yy' = 1 \quad |\div 2y$

$y' = \frac{1}{2y} = \frac{1}{2\sqrt{x}}$

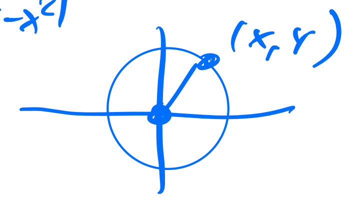
- Example: Do

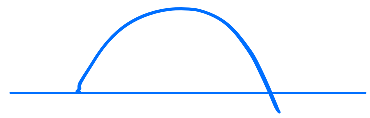
$$f(x) = \sqrt{1 - x^2}$$

explicitly and implicitly

$x^2 + y^2 = 1$

$y = \sqrt{1-x^2}$





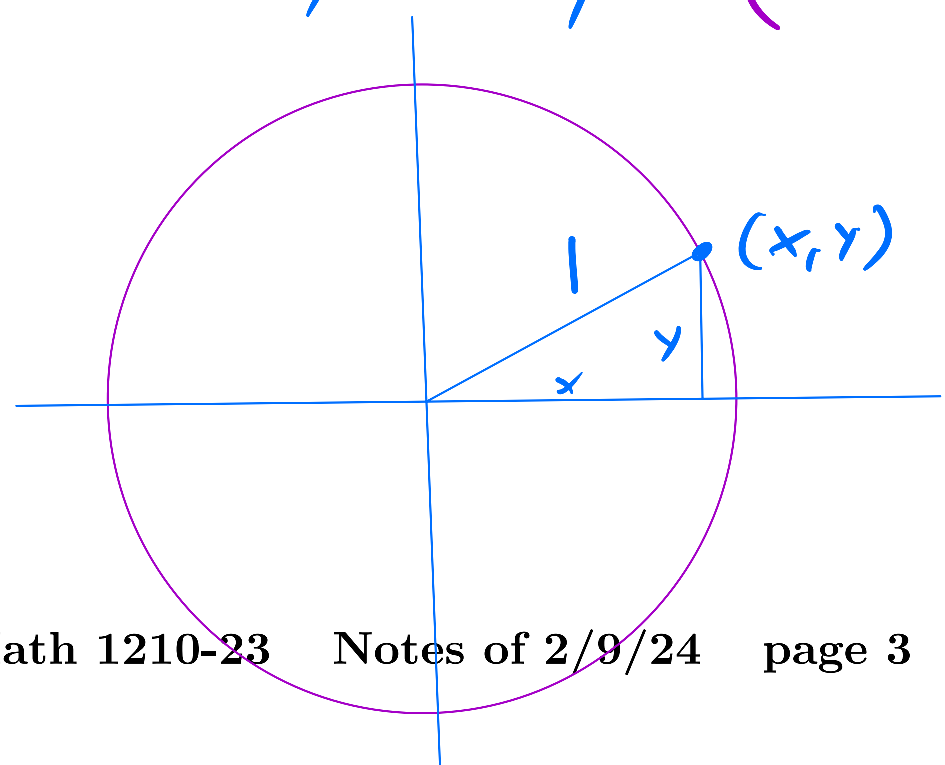
$$\frac{df}{dx} = \frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$$

$$x^2 + y^2 = 1$$

$$2x + 2yy' = 0$$

$$yy' = -x$$

$$y' = -\frac{x}{y} \quad \left(= -\frac{x}{\sqrt{1-x^2}} \right)$$



$x^2 + y^2 = 1$

- Another example:

$$y = y(x), \qquad y^3 + 7y = x^3$$

$x=2 \quad y=1$

$8 = 1 + 7 \cdot 1 = 2^3 = 8$

- We think of y as a function of x
- It's easy to check that the point $(2, 1)$ is on the graph.
- What is $y'(2)$?

$$3y^2 y' + 7y' = 3x^2$$

$$y'(3y^2 + 7) = 3x^2$$

$$y' = \frac{3x^2}{3y^2 + 7}$$

$x = 2$

$y = 1$

$$y' = \frac{3 \cdot 2^2}{3 \cdot 1^2 + 7} = \frac{12}{10} = \frac{6}{5}$$

- We computed the derivative (in terms of x and y) at $x = 2$ without actually having an expression for y in terms of x.
- Pretty Cool!
- It's crucial to **differentiate first and evaluate second**.
- If we were to evaluate at $x = 2$ and $y = 1$ in

$$y^3 + 7y = x^3$$

we'd get

$$8 = 8$$

- Differentiating in that equation gives

$$0 = 0$$

which is true but does not tell us about $y'(2)$.

$$y = f(x)$$
$$y' = f'(x)$$

- Profound application:

$$\frac{\mathrm{d}}{dx}x^r = rx^{r-1}$$

$\frac{d}{dx}x^n = nx^{n-1}$

for all rational numbers r.

- At this stage, technically we know the power rule only if r is an integer.

$r = \frac{p}{q}$ $\quad p, q$ integer

$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{p/q-1}$?

$y = x^{p/q}$

$y^q = x^p \quad |\,(\ )'$

$q\,y^{q-1}y' = p\,x^{p-1}$

$y' = \frac{p\,x^{p-1}}{q\,y^{q-1}}$ $\qquad y = x^{p/q}$

$y^{q-1} = \left(x^{p/q}\right)^{q-1} = x^{\frac{p}{q}(q-1)}$

$= x^{p-\frac{p}{q}}$

$$y' = \frac{p\,x^{p-1}}{q\,x^{p-p/q}}$$

$$= \frac{p}{q}\,x^{p-1-(p-\frac{p}{q})}$$

$$= \frac{p}{q}\,x^{p/q-1}$$

- Let's do another example where we can get the same answer both ways and practice differentiation as well.
- Example 1, simplified:

$$x^2y - y = x^3 - 1$$

$$y(x^2-1) = x^3-1 \quad y = \frac{x^3-1}{x^2-1}$$

implicit

$$2xy + x^2y' - y' = 3x^2$$

$$y'(x^2-1) = 3x^2 - 2xy$$

$$y' = \frac{3x^2 - 2xy}{x^2-1}$$

$$= \frac{3x^2 - 2x\dfrac{x^3-1}{x^2-1}}{x^2-1} \quad \Bigg| \cdot \frac{x^2-1}{x^2-1}$$

$$= \frac{3x^2(x^2-1) - 2x(x^3-1)}{(x^2-1)^2}$$

explicitly

$$\frac{d}{dx} y = \frac{d}{dx} \frac{x^3 - 1}{x^2 - 1}$$

$$= \frac{3x^2(x^2-1) - (x^3-1)\,2x}{(x^2-1)^2}$$

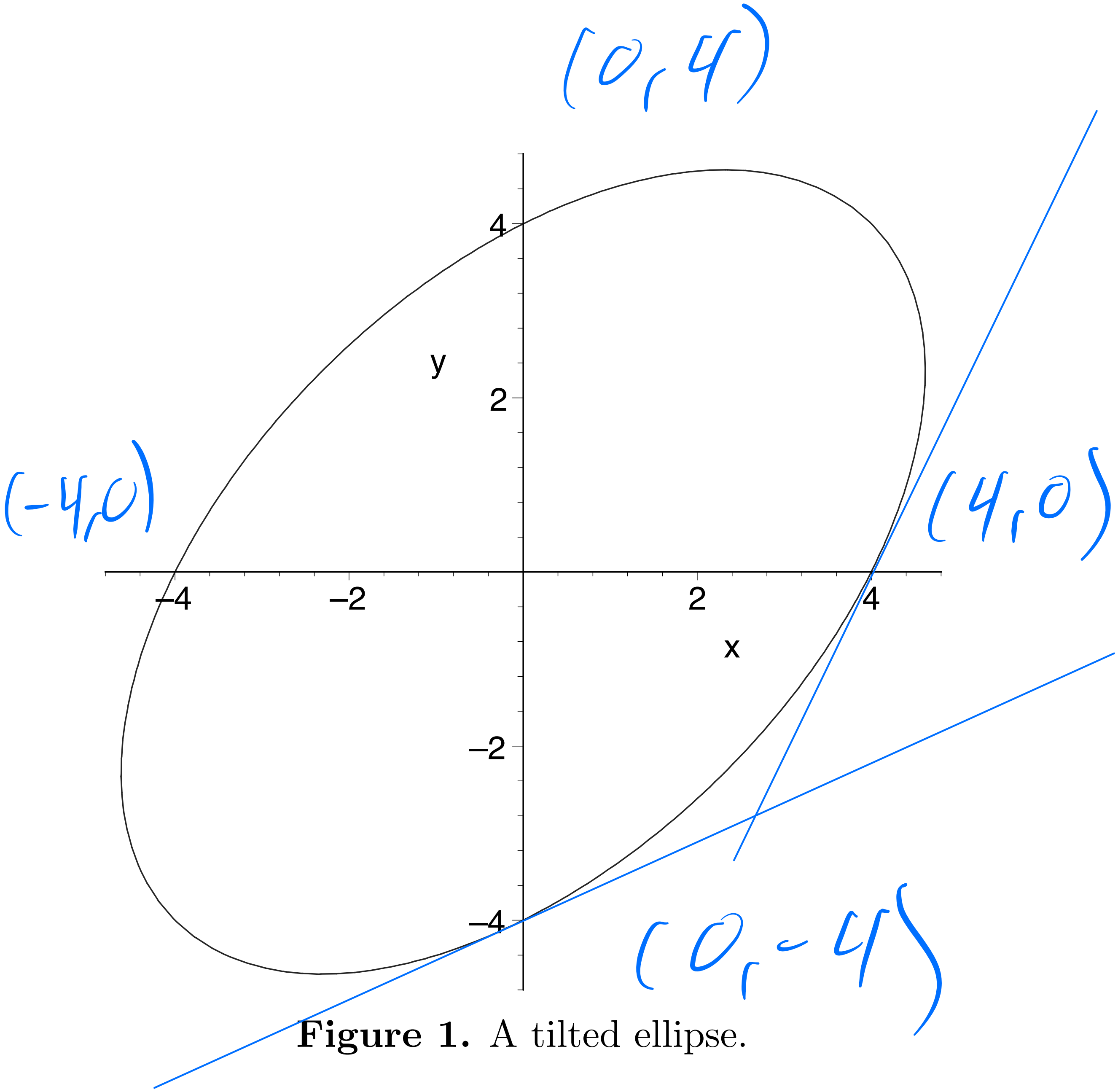


Figure 1. A tilted ellipse.

- Another Example. Problem 47, page 135, textbook. The graph of

$$x^2 - xy + y^2 = 16$$

is a tilted ellipse as shown in Figure 1. Find the equations of the tangents at the x intercepts.

$$2x - (y + xy') + 2yy' = 0$$

$x = 4 \quad y = 0$

$$8 - 4y' = 0$$

$$y' = 2$$

$x = 0 \quad y = 4$

$$-4 + 8y' = 0$$

$$y' = \frac{1}{2}$$