2.6 Higher Order Derivatives

- Recall our differentiation rules:

\[(kf)' = kf'\]  \hspace{1cm} \text{Constant Multiple Rule}
\[(f + g)' = f' + g'\]  \hspace{1cm} \text{Sum Rule}
\[(x^n)' = nx^{n-1}\]  \hspace{1cm} \text{Power Rule (n integer)}
\[(fg)' = f'g + fg'\]  \hspace{1cm} \text{Product Rule}
\[
\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}\]  \hspace{1cm} \text{Quotient Rule}
\[
\frac{d}{dx} \sin x = \cos x
\]  \hspace{1cm} \text{Sine Rule}
\[
\frac{d}{dx} \cos x = -\sin x
\]  \hspace{1cm} \text{Cosine Rule}
\[
\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)
\]  \hspace{1cm} \text{Chain Rule}
\[
\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}\]  \hspace{1cm} \text{Square Root Rule}
• Today’s small extension of what we did previously: Of course we can differentiate more than once!

• Example:

\[ f(x) = 3x^4 - 2x^3 + 5x^2 - 4x + 8 \]
\[ f'(x) = 12x^3 - 6x^2 + 10x - 4 \]
\[ f''(x) = 36x^2 - 12x + 10 \]
\[ f'''(x) = 72x - 12 \]
\[ f^{(4)}(x) = 72 \]
\[ f^{(5)}(x) = 0 \]
Compute the 17-th, 18-th, and n-th derivatives

\[
f^{(17)}(x) = 17 \cdot 16 \cdot 15 \cdots 2 \cdot 1 = 17!
\]

\[
f^{(18)}(x) = 0
\]

\[
f^{(n)}(x) = \begin{cases} 
17 \cdot 16 \cdots & \text{for } n \leq 17, \\
17! & \text{for } n = 17, \\
0 & \text{for } n > 17
\end{cases}
\]

- Differentiating a polynomial reduces the degree by 1.
- The \((n + 1)\)-th derivative of a polynomial of degree \(n\), and all higher derivatives, are zero.
- Polynomials of degree \(n\) are all functions whose \((n + 1)\)-th derivative is zero.
Notation

- See the (incomplete) Table on page 126.
- Suppose
  \[ y = f(x). \]

Then some notations for various derivatives are:

<table>
<thead>
<tr>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>n-th</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y' )</td>
<td>( y'' )</td>
<td>( y''' )</td>
<td>( y^{(n)} )</td>
</tr>
<tr>
<td>( f' )</td>
<td>( f'' )</td>
<td>( f''' )</td>
<td>( f^{(n)} )</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>( f''(x) )</td>
<td>( f'''(x) )</td>
<td>( f^{(n)}(x) )</td>
</tr>
<tr>
<td>( D_x f )</td>
<td>( D^2_x f )</td>
<td>( D^3_x f )</td>
<td>( D^n_x f )</td>
</tr>
<tr>
<td>( \frac{dy}{dx} )</td>
<td>( \frac{d^2y}{dx^2} )</td>
<td>( \frac{d^3y}{dx^3} )</td>
<td>( \frac{d^ny}{dx^n} )</td>
</tr>
</tbody>
</table>
• Another example:

\[ f(x) = \sin x \]

\[
\begin{align*}
    f'(x) &= \cos x \\
    f''(x) &= -\sin x \\
    f'''(x) &= -\cos x \\
    f^{(4)}(x) &= \sin x = f(x)
\end{align*}
\]

\[ f^{(1007)}(x) = -\cos x \]

• The fourth derivative of sine (or cosine) equals the function itself:

\[ y^{(4)} = y \]

• This is our first example of a **Differential Equation**.
Another Example:

\[ y = \sin 2x \]

\[ y' = 2 \cos 2x \]
\[ y'' = -4 \sin 2x \]
\[ y''' = -8 \cos 2x \]
\[ y^{(4)} = 16 \sin 2x \]
\[ y^{(1000)} = 2^{1000} \sin 2x \]
• Recall the height formula from preCalculus:

\[ h(t) = -16t^2 + v_0 t + h_0 \]

where \( v_0 \) is the initial velocity and \( h_0 \) is the initial height.

• The velocity is the derivative of height and acceleration is the derivative of velocity.

\[ v(t) = -32t + v_0 \]
\[ a(t) = -32 \quad \text{[ft/s^2]} \]

• The third derivative of location (height) is sometimes called \textit{jerk}.

• The jerk you feel when coming to a stop in a car is a jump discontinuity in the second derivative.

• Richard Nixon: “Inflation is still increasing. But the rate at which it is increasing is decreasing”.

• What derivative (of cost) is he talking about?
• Differentiating a polynomial simplifies it.

• Differentiating a rational function makes it more complicated.

• Example, compute the first few derivatives of

\[ f(x) = \frac{1}{x^2 + 1} \]

\[ f'(x) = -\frac{2x}{(x^2 + 1)^2} \]

\[ f''(x) = \frac{-2(x^2 + 1)^2 + 2x \cdot 2(1+x^2) \cdot 2x}{(x^2 + 1)^4} \]

\[ = -\frac{2(x^2 + 1) + 8x^2}{(x^2 + 1)^3} \]

\[ = \frac{6x^2 - 2}{(x^2 + 1)^3} \]
\[ f'''(x) = \frac{12x(x^2+1)^3 - (6x^2-2) \cdot 3(x^2+1)^2 \cdot 2x}{(x^2+1)^6} \]

\[ = \frac{12x(x^2+1) - (6x^2-2) \cdot 6x}{(x^2+1)^4} \]
• Sometimes we can simplify differentiation with suitable tricks.

• Example: Compute the fourth derivative of

\[
f(x) = \frac{x^4 + x^3 + x^2 + x + 1}{x}
\]

\[
x^{-1}(x^4 + x^3 + x^2 + x + 1)
\]

\[
= (x^3 + x^2 + x + 1) + x^{-1}
\]

\[
\frac{d}{dx} -x^{-2} \Rightarrow 2x^{-3} \Rightarrow -6x^{-4} \Rightarrow 24x^{-5} \Rightarrow \frac{24}{x^5}
\]

• What about

\[
f(x) = \frac{x^4 + x^3 + x^2 + x + 1}{x-1}
\]

(exercise!)
• Differentiation also tends to make trigonometric functions more complicated.

• Example:

\[ f(x) = \tan x = \frac{\sin x}{\cos x}. \]

\[ f'(x) = \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \]

\[ f''(x) = \frac{2 \cos x \sin x}{\cos^4 x} = \frac{2 \sin x}{\cos^3 x} \]

\[ f'''(x) = \frac{2 \cos x \cdot \cos^3 x + 2 \sin x \cdot 3 \cos^2 x}{\cos^6 x} \]
\[
\frac{2 \cos^2 x + 6 \sin^2 x}{\cos^4 x}
\]

\[
\lim_{n \to \infty} \frac{\sin x}{n} = \lim_{n \to \infty} \frac{\sin \frac{x}{n}}{\frac{x}{n}} = \lim_{u \to 0} \frac{\sin u}{u} = 0
\]