Math 1210-23

Notes of 2/7/24

2.6 Higher Order Derivatives

- Recall our differentiation rules:

$(kf)'$	$=$	kf'	**Constant Multiple Rule**
$(f+g)'$	$=$	$f'+g'$	**Sum Rule**
$(x^n)'$	$=$	nx^{n-1}	**Power Rule (n integer)**
$(fg)'$	$=$	$f'g+fg'$	**Product Rule**
$\left(\frac{f}{g}\right)'$	$=$	$\frac{f'g-fg'}{g^2}$	**Quotient Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\sin x$	$=$	$\cos x$	**Sine Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\cos x$	$=$	$-\sin x$	**Cosine Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}f\big(g(x)\big)$	$=$	$f'\big(g(x)\big)g'(x)$	**Chain Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}$	$=$	$\frac{1}{2\sqrt{x}}$	**Square Root Rule**

- Today's small extension of what we did previously: Of course we can differentiate more than once!
- Example:

$$f(x) = 3x^4 - 2x^3 + 5x^2 - 4x + 8$$

$$f'(x) = 12x^3 - 6x^2 + 10x - 4$$

$$f''(x) = 36x^2 - 12x + 10$$

$$f'''(x) = 72x - 12$$

$$f^{IV}(x) = 72$$

$$f^{V}(x) = 0$$

$(n+1)-th$

$$f(x) = x^{17}$$

- Compute the 17-th, 18-th, and n-th derivatives

$$f^{(17)}(x) = 17 \cdot 16 \cdot 15 \cdots 2 \cdot 1 = 17!$$

$$f^{(18)}(x) = 0$$

$$f^{(n)}(x) = \begin{cases} 17 \cdot 16 \cdots \quad x^{17-n+1} & n < 17 \\ 17! & n = 17 \\ 0 & n > 17 \end{cases}$$

- Differentiating a polynomial reduces the degree by 1.
- The $(n+1)$-th derivative of a polynomial of degree n, and all higher derivatives, are zero.
- Polynomials of degree n are all functions whose $(n+1)$-th derivative is zero.

Notation

- See the (incomplete) Table on page 126.
- Suppose
$$y = f(x).$$
Then some notations for various derivatives are:

1^{st}	y'	f'	$f'(x)$	$D_x f$	$\frac{\mathrm{d}y}{\mathrm{d}x}$
2^{nd}	y''	f''	$f''(x)$	$D_x^2 f$	$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$
3^{rd}	y'''	f'''	$f'''(x)$	$D_x^3 f$	$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3}$
n-th	$y^{(n)}$	$f^{(n)}$	$f^{(n)}(x)$	$D_x^n f$	$\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$

- Another example:

$$f(x) = \sin x$$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x$$

$$f^{IV}(x) = \sin x = f(x)$$

$$f^{(1007)}(x) = -\cos x$$

- The fourth derivative of sine (or cosine) equals the function itself:

$$y^{(4)} = y$$

- This is our first example of a **Differential Equation**.

- Another Example:

$$y = \sin 2x$$

$$y' = 2\cos 2x$$

$$y'' = -4\sin 2x$$

$$y''' = -8\cos 2x$$

$$y^{\overline{\underline{IV}}} = 16\sin 2x$$

$$y^{(1000)} = 2^{1000}\sin 2x$$

- Recall the height formula from preCalculus:

$$h(t) = -16t^2 + v_0 t + h_0$$

  where v_0 is the initial velocity and h_0 is the initial height.

- The velocity is the derivative of height and acceleration is the derivative of velocity.

jerk

$$v(t) = -32t + v_0$$

$$a(t) = -32 \qquad [f/s^2]$$

- The third derivative of location (height) is sometimes called **jerk**.

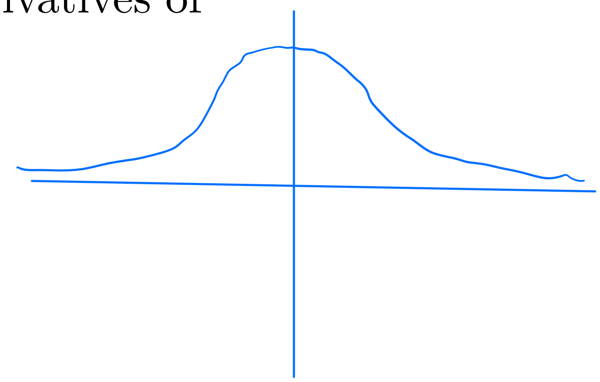
- The jerk you feel when coming to a stop in a car is a jump discontinuity in the second derivative.

- Richard Nixon: "Inflation is still increasing. But the rate at which it is increasing is decreasing".

- What derivative (of cost) is he talking about?

- Differentiating a polynomial simplifies it.
- Differentiating a rational function makes it more complicated.
- Example, compute the first few derivatives of

$$f(x) = \frac{1}{x^2+1}$$



$$f'(x) = \frac{-2x}{(x^2+1)^2}$$

$$f''(x) = \frac{-2(x^2+1)^2 + 2x \cdot 2(1+x^2) \cdot 2x}{(x^2+1)^4}$$

$$= \frac{-2(x^2+1) + 8x^2}{(x^2+1)^3}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$= \frac{6x^2 - 2}{(x^2+1)^3}$$

$$f'''(x) = \frac{12x(x^2+1)^3 - (6x^2-2)\cdot 3(x^2+1)^2\cdot 2x}{(x^2+1)^6}$$

$$= \frac{12x(x^2+1) - (6x^2-2)\cdot 6x}{(x^2+1)^4}$$

- Sometimes we can simplify differentiation with suitable tricks.
- Example: Compute the fourth derivative of

$$f(x) = \frac{x^4 + x^3 + x^2 + x + 1}{x} = x^3 + x^2 + x + 1 + \frac{1}{x}$$

$$x^{-1}(x^4 + x^3 + x^2 + x + 1)$$

$$= (x^3 + x^2 + x + 1) + x^{-1}$$

$$x^{-1} \xrightarrow[\frac{d}{dx}]{} -x^{-2} \to 2x^{-3} \to -6x^{-4} \to 24x^{-5} = \frac{24}{x^5}$$

$$= \frac{4!}{x^5}$$

$$\begin{array}{r|l} & x^3 + 2x^2 + 3x + 4 \\ \hline (x-1) & x^4 + x^3 + x^2 + x + 1 \\ & \underline{x^4 - x^3} \\ & 2x^3 \\ & \underline{2x^3 - 2x^2} \\ & 3x^2 \\ & \underline{3x^2 - 3x} \\ & 4x + 1 \\ & \underline{4x - 4} \\ & 5 \end{array}$$

- What about

$$f(x) = \frac{x^4 + x^3 + x^2 + x + 1}{x - 1}$$

(exercise!)

$$f(x) = x^3 + 2x^2 + 3x + 4 + \frac{5}{x-1}$$

- Differentiation also tends to make trigonometric functions more complicated.
- Example:

$$f(x) = \tan x = \frac{\sin x}{\cos x}.$$

$$f'(x) = \frac{d}{dx} \frac{\sin x}{\cos x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$f''(x) = \frac{+2 \cos x \sin x}{\cos^4 x}$$

$$= \frac{2 \sin x}{\cos^3 x}$$

$$f'''(x) = \frac{2 \cos x \cdot \cos^3 x + 2 \sin^2 x \cdot 3 \cos^2 x}{\cos^6 x}$$

$$= \frac{2\cos^2 x + 6\sin^2 x}{\cos^4 x}$$

$$\lim_{n\to\infty} \frac{\sin x}{n} = \lim_{n\to\infty} \frac{\mathrm{si}\cancel{n}x}{\cancel{n}} = \lim_{n\to\infty} \mathrm{six} = 6$$