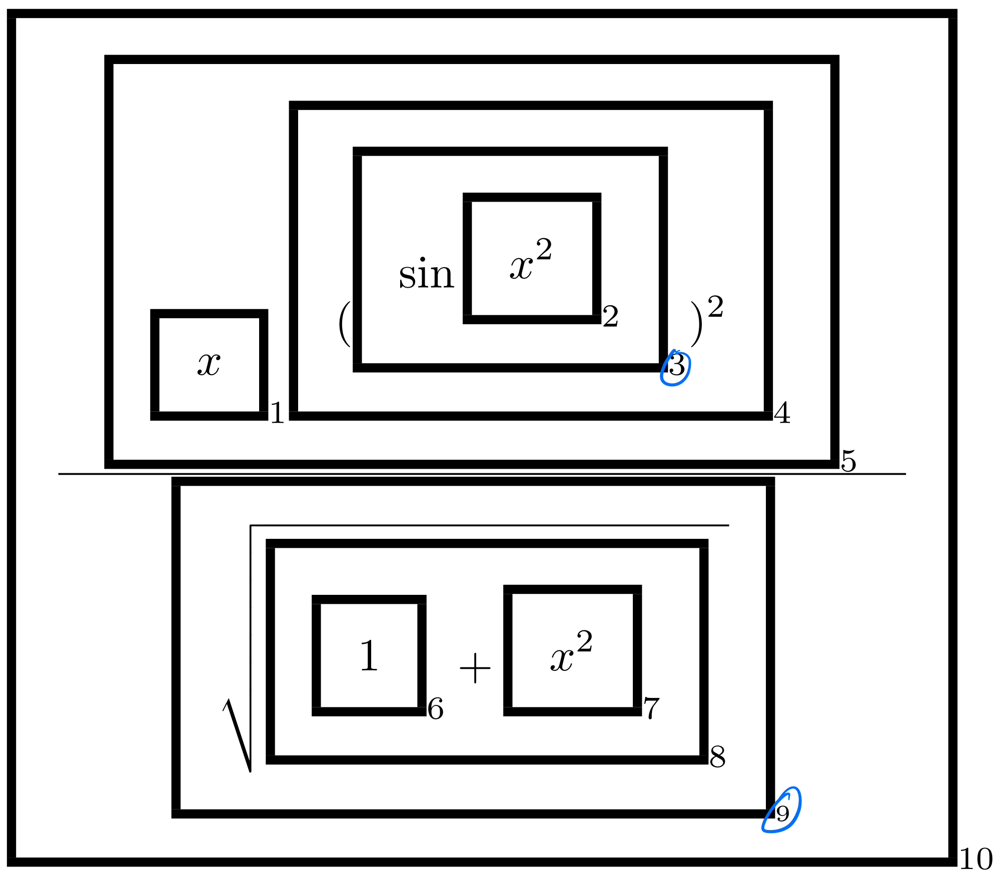
Math 1210-23, Spring 2024

Notes of 2/6/24

The Onion Method of Differentiation

$$f(x) = \frac{x \sin^2 x^2}{\sqrt{1+x^2}} =$$



$$f'(x) = \frac{\left(\sin^2 x^2 + x \cdot 2 \sin x^2 \cos x^2 \cdot 2x\right)\sqrt{1+x^2} - x \sin^2 x^2 \cdot \frac{2x}{2\sqrt{1+x^2}}}{1+x^2}$$

- Let's note what rule to apply:

1. Pow R
2. Pow R
3. Sin R
4. Pow R
5. P R
6. const R
7. Pow R
8. Sum R
9. $\sqrt{}$ R
10. Q R

Now compute the derivative, and mark it with square brackets if you like. This is problem 19 of hw 5. It will take you some time to work out the derivative, and to write it in a form that ww will interpret correctly.

- Of course, you don't want to have to draw boxes and label them.
- Instead handle the onion layers mentally.
- Problem 64, page 124. Find the equation of the tangent line to

$$y = (x^2+1)^3(x^4+1)^2 \quad = f(x) \qquad (0, 1)$$

at $(1, 32)$. ✓

$$x = 1 \quad y = (1^2+1)^3(1^4+1)^2$$

$$= 2^3\, 2^2 = 2^5 = 32$$

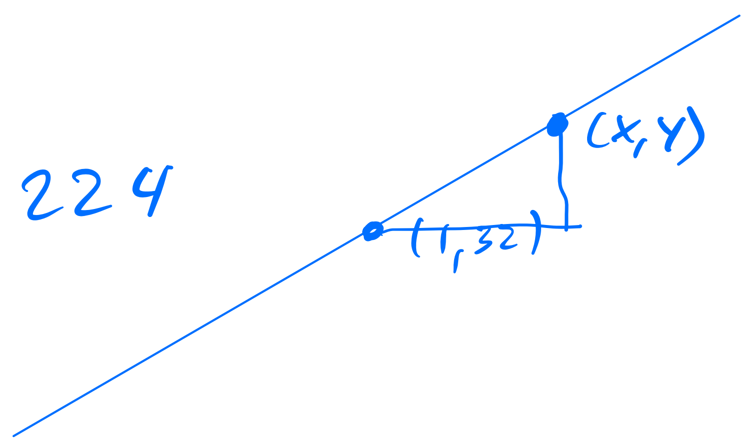
$$y' = 3(x^2+1)^2\, 2x\, (x^4+1)^2 + (x^2+1)^3\, 2(x^4+1)\, 4x^3$$

$$x = 1 \quad y' = 3 \cdot 2^2 \cdot 2 \cdot 2^2 + 2^3 \cdot 2 \cdot 2 \cdot 4$$

$$= 3 \cdot 2^5 + 4 \cdot 2^5$$

$$= 3 \cdot 32 + 4 \cdot 32$$

$$= 96 + 128 = 224$$



$$\frac{y-32}{x-1} = 224$$

$$= 224(x-1)$$

$$y - 32 = 224x - 224 \quad | +32$$

$$y = 224x - 192$$

- A major way to check your answers is to compute them in two different ways. For example, problem 14, page 123. Compute

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-2}{x-3}\right)^{-3} = \frac{d}{dx}\frac{(x-3)^3}{(x-2)^3} \qquad x^{-4} = \frac{1}{x^4}$$

in two (or more) different ways:

$$f'(x) = -3\left(\frac{x-2}{x-3}\right)^{-4}\frac{x-3-(x-2)}{(x-3)^2}$$

$$= +3\,\frac{(x-3)^4}{(x-2)^4}\,\frac{1}{(x-3)^2}$$

$$= \frac{3(x-3)^2}{(x-2)^4}$$

$$= \frac{d}{dx}\frac{(x-3)^3}{(x-2)^3} = \frac{3(x-3)^2(x-2)^3 - (x-3)^3\cdot 3(x-2)^2}{(x-2)^6}$$

$$(x-3)^3 = (x-3)(x-3)^2$$

$$= \frac{3((x-3)^2(x-2) - (x-3)^3)}{(x-2)^4}$$

$$= \frac{3(x-3)^2(x-2-(x-3))}{(x-2)^4}$$

$$= \frac{3(x-3)^2}{(x-2)^4}$$

- Problem 81, page 125. Suppose $f(0) = 0$ and $f'(0) = 2$. What is

$$\frac{\mathrm{d}}{\mathrm{d}x} f(f(\underline{f(f(x)}))))$$

at $x = 0$? What about

$$\frac{\mathrm{d}}{\mathrm{d}x} \qquad \underbrace{f(f(f}_{\text{100 evaluations of } f} \qquad \cdots \qquad \underbrace{)\ldots)}_{\text{100 closing parentheses}} \qquad ?$$

$f(0) = 0$

$f'(0) = 2$

$$\frac{d}{dx} f(f(x)) = \underbrace{f'(\underbrace{f(x)}_{0})}_{2} \cdot \underbrace{f'(x)}_{2}$$

$x = 0$ $\qquad$ 4

$$\frac{d}{dx} f(f(f(x))) = \underbrace{f'(\underbrace{f(f(x))}_{0})}_{2} \cdot \frac{d}{dx} f(f(x)) \qquad 8$$

$x = 0$

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$$\frac{d}{dx} f(f(f(f(x))))$$

$x=0$

- Use the product rule to derive the power rule for positive integers by induction.

- Reinterpret the quotient rule by applying the power rule: how

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\times\frac{1}{g(x)}\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}f(x)\times\frac{1}{g(x)} + f(x)\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(x)}\\
&= \frac{f'(x)}{g(x)} + f(x)\times\left(-\frac{g'(x)}{g^2(x)}\right)\\
&= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g^2(x)}\\
&\qquad\qquad \text{which is of course the same as}\\
&= \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}
\end{aligned}$$

Higher Order Derivatives

- Differentiating a function gives a new function. How about differentiating that new function?
- For example, the derivative of location is velocity, and the derivative of velocity is acceleration!
- acceleration is the second derivative of location.
- Higher order derivatives are our next subject.