Math 1210-3 Notes of 2/5/24

More Differentiation

- Recall our differentiation rules:

$(kf)'$	$=$	kf'	**Constant Multiple Rule**
$(f+g)'$	$=$	$f'+g'$	**Sum Rule**
$(x^n)'$	$=$	nx^{n-1}	**Power Rule**
$(fg)'$	$=$	$f'g+fg'$	**Product Rule**
$\left(\frac{f}{g}\right)'$	$=$	$\frac{f'g-fg'}{g^2}$	**Quotient Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\sin x$	$=$	$\cos x$	**Sine Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\cos x$	$=$	$-\sin x$	**Cosine Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}f\big(g(x)\big)$	$=$	$f'\big(g(x)\big)g'(x)$	**Chain Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}$	$=$	$\frac{1}{2\sqrt{x}}$	**Square Root Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x}$	$=$	$-\frac{1}{x^2}$	**Reciprocal Rule**

- We can differentiate many functions by a combination of these rules.

$$\frac{d}{dx}\frac{x+1}{x^2+1} = \frac{x^2+1-(x+1)2x}{(x^2+1)^2}$$

$= g(x)$

$$= \frac{-x^2-2x+1}{(x^2+1)^2}$$

$$= \frac{d}{dx}(x+1)(x^2+1)^{-1} = (x^2+1)^{-1} + (x+1)(-1)(x^2+1)^{-2}(2x)$$

$$= \frac{1}{x^2+1} - \frac{2x(x+1)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2-2x}{(x^2+1)^2}$$

$$= \frac{-x^2-2x+1}{(x^2+1)^2} \checkmark$$

$f(x) = x^{100}$

$$\frac{d}{dx}\left(\frac{x+1}{x^2+1}\right)^{100} =$$

$$= 100\left(\frac{x+1}{x^2+1}\right)^{99} \cdot \frac{-x^2-2x+1}{(x^2+1)^2}$$

$$\frac{d}{dx}\sin\left(\frac{x+1}{x^2+1}\right)^{100} = \cos\left(\frac{x+1}{x^2+1}\right)^{100} \cdot$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x+\sin x)^{100} = 100(x+\sin x)^{99}(1+\cos x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\cos^2 x}{2+\sin x^2} = \frac{-2\cos x\sin x(2+\sin x^2) - \cos^2 x\cdot\cos x^2\cdot 2x}{(2+\sin x^2)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)\big)^n = n\big(f(x)\big)^{n-1}\cdot f'(x)$$

$f(x)=x^n \quad \frac{d}{dx}\big(f(x)\big)^n = n\big(f(x)\big)^{n-1} f'(x)$

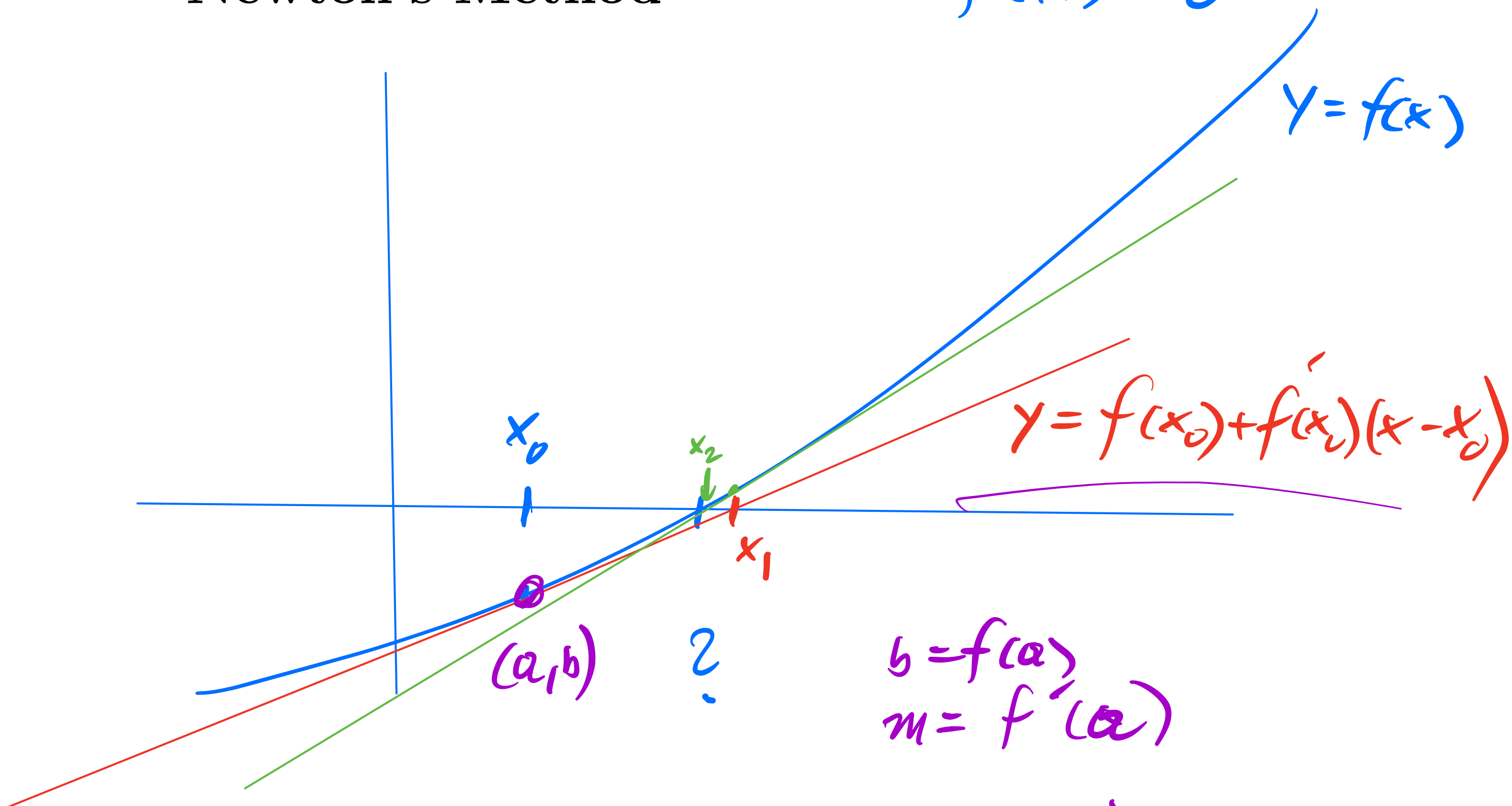
$= n x^{n-1}\cdot 1$

$= n x^{n-1}$

- This last formula is called the general (or generalized) power rule.

Newton's Method

$f(x)=0$



$b=f(a)$

$m=f'(a)$

$$y=b+m(x-x_0)$$
$$=mx+(b-mx_0)$$
$$=mx+c$$
$$y(x_0)=b=f(x_0)$$

$$y=f(x_0)+f'(x_0)(x-x_0)=0$$
$$f'(x_0)(x-x_0)=-f(x_0)$$
$$x-x_0=-\frac{f(x_0)}{f'(x_0)}$$
$$x_1=x=x_0-\frac{f(x_0)}{f'(x_0)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0, 1, 2, 3, \ldots$$

Example: approximate $\sqrt{2}$ by applying Newton's Method to

$$f(x) = x^2 - 2 = 0. \qquad x = \sqrt{2}$$

$$f'(x) = 2$$

$x_0 = 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}$$

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    |\^/|     Maple 2016 (X86 64 LINUX)
._|\|   |/|_. Copyright (c) Maplesoft,
a division of Waterloo Maple Inc. 2016
 \  MAPLE  /  All rights reserved. Maple
is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.
> restart:
> Digits:=50:
> f:=x^2-2:
> g:=x-f/diff(f,x):
> xn:=1:
> lprint(sqrt(2.0)):
1.4142135623730950488016887242096980785696718753769
>
> for i from 1 to 8 do
>    xn:=evalf(subs(x=xn,g)):
>    lprint(i,xn):
>    end do:
1, 1.5000000000000000000000000000000000000000000000000
2, 1.4166666666666666666666666666666666666666666666667
3, 1.4142156862745098039215686274509803921568627450981
4, 1.4142135623746899106262955788901349101165596221157
5, 1.4142135623730950488016896235025302436149819257762
6, 1.4142135623730950488016887242096980785696718753772
7, 1.4142135623730950488016887242096980785696718753770
8, 1.4142135623730950488016887242096980785696718753770
>
> quit
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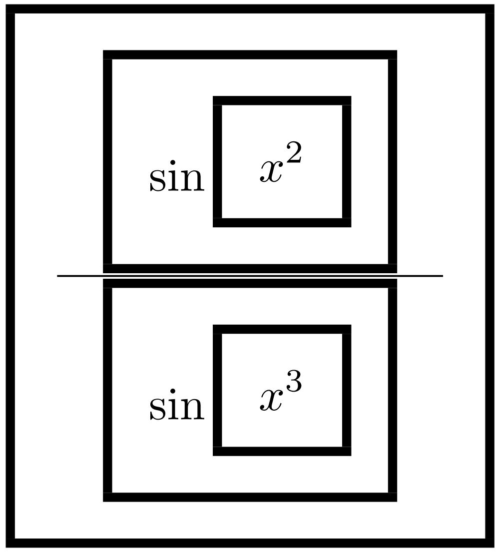
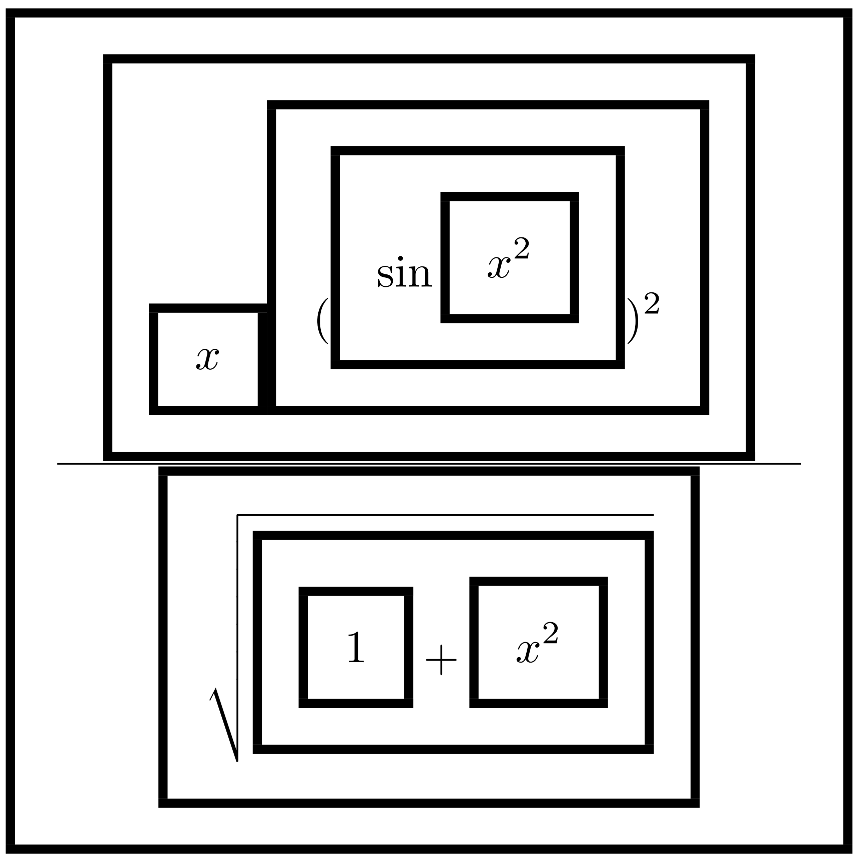
The Onion Method of Differentiation

- **Onion Method:** Apply the rule that is appropriate for the last operation needed to **evaluate** the expression. Repeat as needed for the ingredients of that expression.
- Examples:

$$f(x) = \frac{\sin x^2}{\sin x^3} \quad =$$



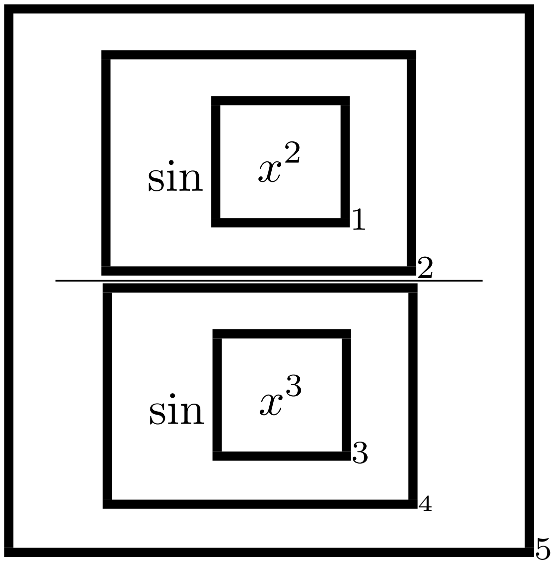
$$f(x) = \frac{x \sin^2 x^2}{\sqrt{1+x^2}} \quad =$$



- Note: The second function is the subject of problem 19 on hw 5.

- The first example has 5 boxes. Let's label them and state what rules they require:

$$f(x) = \frac{\sin x^2}{\sin x^3} =$$



1. PR
2. sin
3. PR
4. sin
5. QR

Let's do the actual differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x^2}{\sin x^3} = \frac{\cos x^2 \cdot 2x \sin x^3 - \sin x^2 \cos x^3 \cdot 3x^2}{\sin^2 x^3}$$

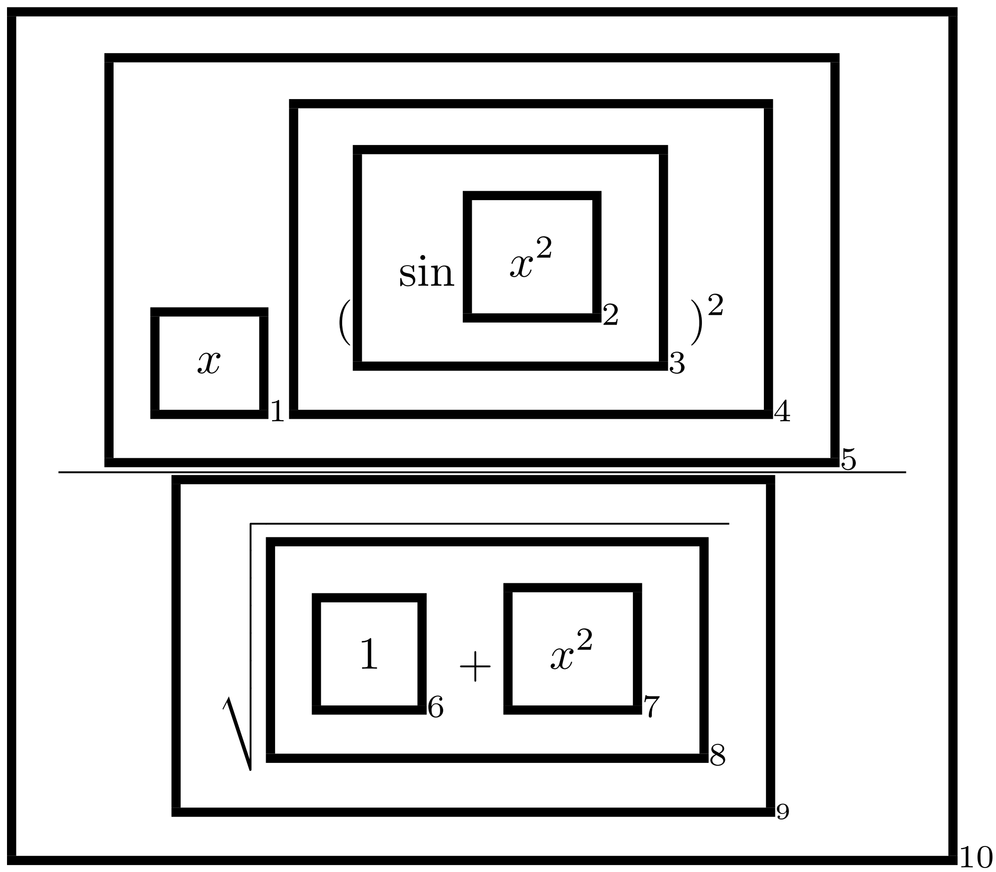
- Here is the same result with the derivatives of the boxed terms indicated by square brackets

$$f(x) = \frac{\sin x^2}{\sin x^3} = \boxed{\frac{\boxed{\sin \boxed{x^2}_1}_2}{\boxed{\sin \boxed{x^3}_3}_4}}_5$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x^2}{\sin x^3} = \left[\frac{\left[\cos x^2 \, [2x]_1\right]_2 \sin x^3 - \sin x^2 \left[\cos x^3 \, \left[3x^2\right]_3\right]_4}{\sin^2 x^3}\right]_5$$

- **Exercise:** do the same with example 2:

$$f(x) = \frac{x \sin^2 x^2}{\sqrt{1+x^2}} =$$



1.
2.
3.
4.
5.
6.
7.
8.
9.
10.

Now compute the derivative, and mark it with square brackets if you like.

- Of course, you don't want to have to draw boxes and label them.
- Instead handle the onion layers mentally.
- Problem 64, page 124. Find the equation of the tangent line to

$$y = (x^2 + 1)^3 (x^4 + 1)^2$$

at $(1, 32)$.

- Problem 81, page 125. Suppose $f(0) = 0$ and $f'(0) = 2$. What is

$$\frac{\mathrm{d}}{\mathrm{d}x} f(f(f(f(x))))$$

at $x = 0$?

- Use the product rule to derive the power rule for positive integers by induction.