

Math 1210-3 Notes of 2/5/24

More Differentiation

- Recall our differentiation rules:

$$(kf)' = kf' \qquad \text{Constant Multiple Rule}$$

$$(f + g)' = f' + g' \qquad \text{Sum Rule}$$

$$(x^n)' = nx^{n-1} \qquad \text{Power Rule}$$

$$(fg)' = f'g + fg' \qquad \text{Product Rule}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \qquad \text{Quotient Rule}$$

$$\frac{d}{dx} \sin x = \cos x \qquad \text{Sine Rule}$$

$$\frac{d}{dx} \cos x = -\sin x \qquad \text{Cosine Rule}$$

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \qquad \text{Chain Rule}$$

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}} \qquad \text{Square Root Rule}$$

$$\frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2} \qquad \text{Reciprocal Rule}$$

- We can differentiate many functions by a combination of these rules.

$$\frac{d}{dx} \frac{x+1}{x^2+1} =$$

$$\frac{d}{dx} \left(\frac{x+1}{x^2+1} \right)^{100} =$$

$$\frac{d}{dx} \sin \left(\frac{x+1}{x^2+1} \right)^{100} =$$

$$\frac{d}{dx}(x + \sin x)^{100} =$$

$$\frac{d}{dx} \frac{\cos^2 x}{2 + \sin x^2} =$$

$$\frac{d}{dx} (f(x))^n =$$

- This last formula is called the general (or generalized) power rule.

Newton's Method

Example: approximate $\sqrt{2}$ by applying Newton's Method to

$$f(x) = x^2 - 2 = 0.$$

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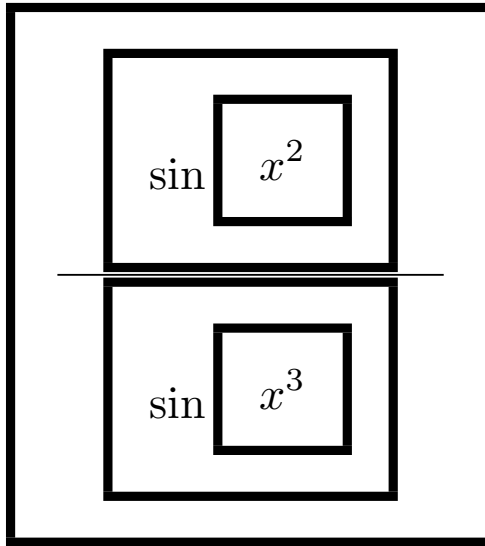
1      |\~/|      Maple 2016 (X86 64 LINUX)
2  ._|\|      |/_|. Copyright (c) Maplesoft,
a division of Waterloo Maple Inc. 2016
3  \ MAPLE / All rights reserved. Maple
is a trademark of
4  <-----> Waterloo Maple Inc.
5      |      Type ? for help.
6  > restart:
7  > Digits:=50:
8  > f:=x^2-2:
9  > g:=x-f/diff(f,x):
10 > xn:=1:
11 > lprint(sqrt(2.0)):
12 1.4142135623730950488016887242096980785696718753769
13 >
14 > for i from 1 to 8 do
15 >     xn:=evalf(subs(x=xn,g)):
16 >     lprint(i,xn):
17 >     end do:
18 1, 1.50000000000000000000000000000000000000000000000000000000000000
19 2, 1.41666666666666666666666666666666666666666666666666666666666667
20 3, 1.4142156862745098039215686274509803921568627450981
21 4, 1.4142135623746899106262955788901349101165596221157
22 5, 1.4142135623730950488016896235025302436149819257762
23 6, 1.4142135623730950488016887242096980785696718753772
24 7, 1.4142135623730950488016887242096980785696718753770
25 8, 1.4142135623730950488016887242096980785696718753770
26 >
27 > quit
28

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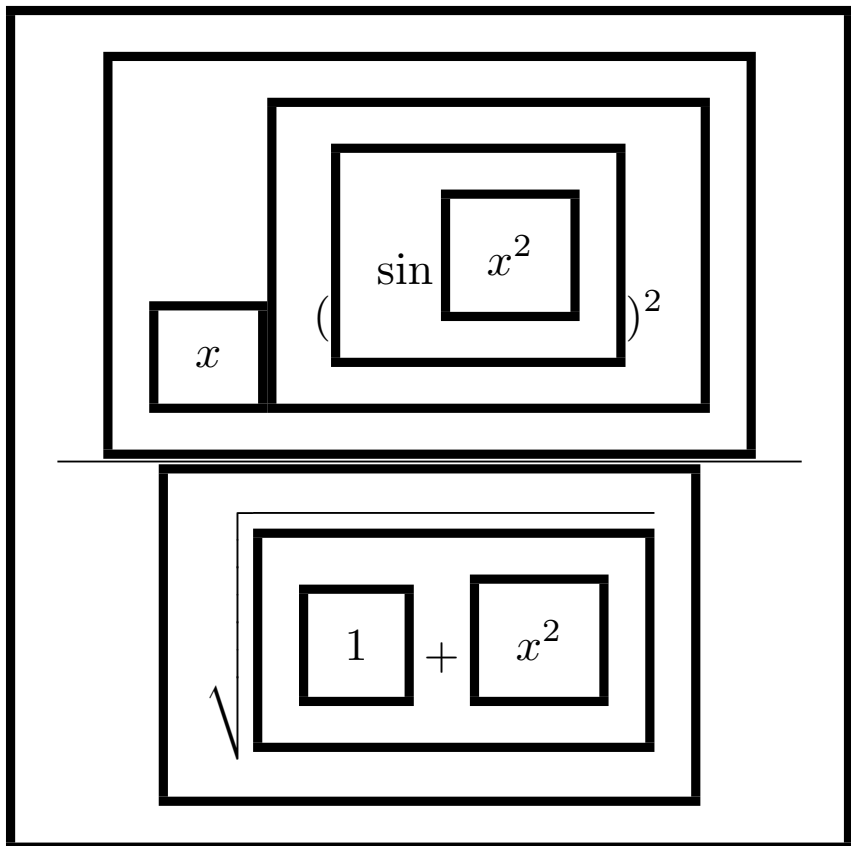
The Onion Method of Differentiation

- **Onion Method:** Apply the rule that is appropriate for the last operation needed to **evaluate** the expression. Repeat as needed for the ingredients of that expression.
- Examples:

$$f(x) = \frac{\sin x^2}{\sin x^3} =$$



$$f(x) = \frac{x \sin^2 x^2}{\sqrt{1+x^2}} =$$



- Note: The second function is the subject of problem 19 on hw 5.

- The first example has 5 boxes. Let's label them and state what rules they require:

$$f(x) = \frac{\sin x^2}{\sin x^3} =$$

- 1.
- 2.
- 3.
- 4.
- 5.

Let's do the actual differentiation:

$$\frac{d}{dx} \frac{\sin x^2}{\sin x^3} =$$

- Here is the same result with the derivatives of the boxed terms indicated by square brackets

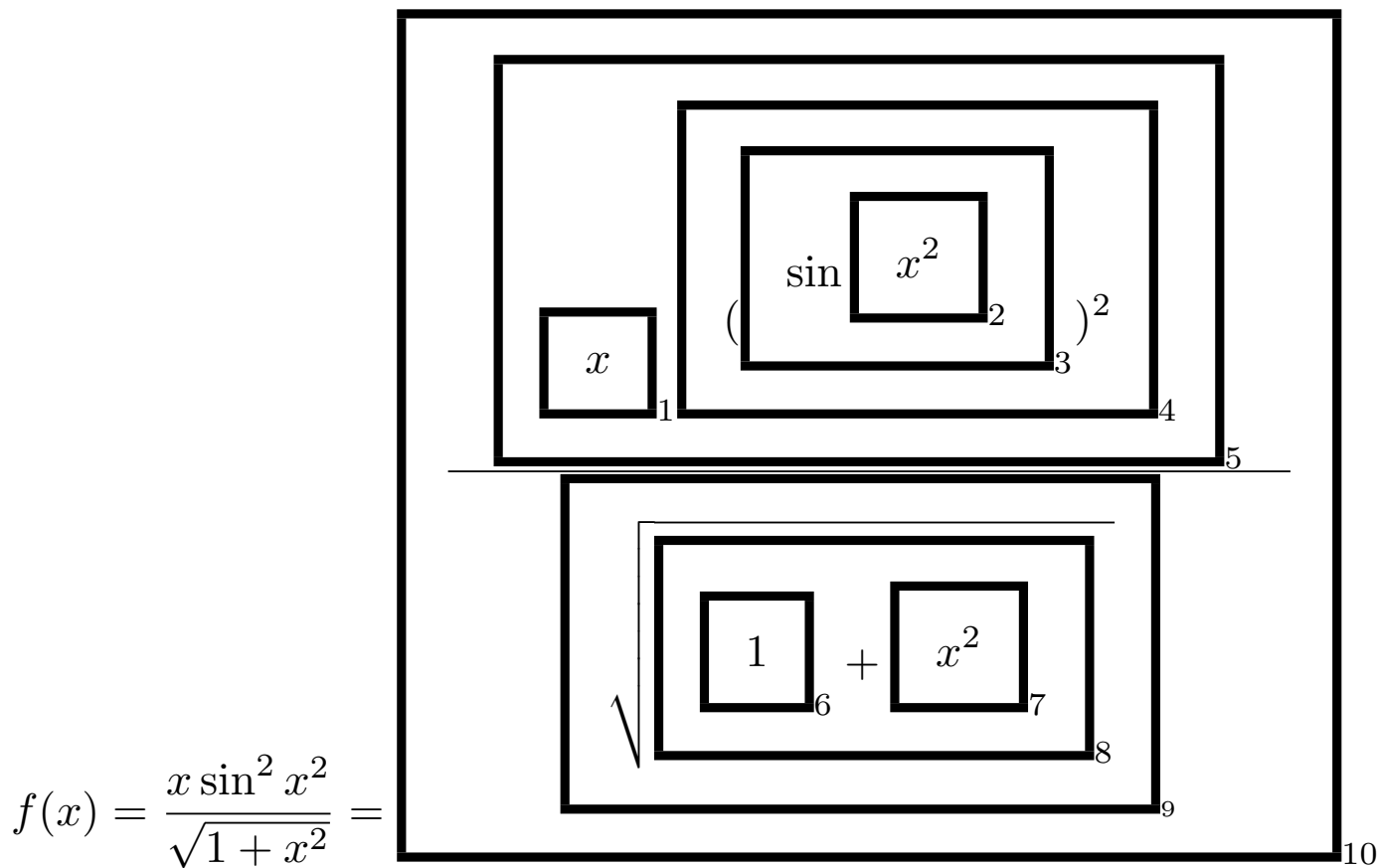
$$f(x) = \frac{\sin x^2}{\sin x^3} = \frac{\sin x^2}{\sin x^3}$$

The diagram shows the function $f(x) = \frac{\sin x^2}{\sin x^3}$ with nested boxes and numbers indicating the order of differentiation for each part of the expression:

- 1: Box around x^2 in the numerator.
- 2: Box around the entire numerator $\sin x^2$.
- 3: Box around x^3 in the denominator.
- 4: Box around the entire denominator $\sin x^3$.
- 5: Box around the entire fraction $\frac{\sin x^2}{\sin x^3}$.

$$\frac{d}{dx} \frac{\sin x^2}{\sin x^3} = \left[\frac{[\cos x^2 [2x]_1]_2 \sin x^3 - \sin x^2 [\cos x^3 [3x^2]_3]_4}{\sin^2 x^3} \right]_5$$

- **Exercise:** do the same with example 2:



- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.

Now compute the derivative, and mark it with square brackets if you like.

- Of course, you don't want to have to draw boxes and label them.
- Instead handle the onion layers mentally.
- Problem 64, page 124. Find the equation of the tangent line to

$$y = (x^2 + 1)^3(x^4 + 1)^2$$

at $(1, 32)$.

- Problem 81, page 125. Suppose $f(0) = 0$ and $f'(0) = 2$. What is

$$\frac{d}{dx} f(f(f(f(x))))$$

at $x = 0$?

- Use the product rule to derive the power rule for positive integers by induction.