# Math 1210-3 Notes of 2/5/24

## More Differentiation

- Recall our differentiation rules:

| | | | |
|---|---|---|---|
| $(kf)'$ | $=$ | $kf'$ | **Constant Multiple Rule** |
| $(f+g)'$ | $=$ | $f'+g'$ | **Sum Rule** |
| $(x^n)'$ | $=$ | $nx^{n-1}$ | **Power Rule** |
| $(fg)'$ | $=$ | $f'g+fg'$ | **Product Rule** |
| $\left(\frac{f}{g}\right)'$ | $=$ | $\frac{f'g-fg'}{g^2}$ | **Quotient Rule** |
| $\frac{\mathrm{d}}{\mathrm{d}x}\sin x$ | $=$ | $\cos x$ | **Sine Rule** |
| $\frac{\mathrm{d}}{\mathrm{d}x}\cos x$ | $=$ | $-\sin x$ | **Cosine Rule** |
| $\frac{\mathrm{d}}{\mathrm{d}x}f\big(g(x)\big)$ | $=$ | $f'\big(g(x)\big)g'(x)$ | **Chain Rule** |
| $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}$ | $=$ | $\frac{1}{2\sqrt{x}}$ | **Square Root Rule** |
| $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x}$ | $=$ | $-\frac{1}{x^2}$ | **Reciprocal Rule** |

- We can differentiate many functions by a combination of these rules.

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x+1}{x^2+1} =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x+1}{x^2+1}\right)^{100} =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin\left(\frac{x+1}{x^2+1}\right)^{100} =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x+\sin x)^{100} =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\cos^2 x}{2+\sin x^2} =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)\big)^n =$$

- This last formula is called the general (or generalized) power rule.

## Newton's Method

Example: approximate $\sqrt{2}$ by applying Newton's Method to

$$f(x) = x^2 - 2 = 0.$$

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    |\^/|     Maple 2016 (X86 64 LINUX)
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 \  MAPLE  /  All rights reserved. Maple
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      |       Type ? for help.
> restart:
> Digits:=50:
> f:=x^2-2:
> g:=x-f/diff(f,x):
> xn:=1:
> lprint(sqrt(2.0)):
1.4142135623730950488016887242096980785696718753769
>
> for i from 1 to 8 do
>    xn:=evalf(subs(x=xn,g)):
>    lprint(i,xn):
>    end do:
1, 1.5000000000000000000000000000000000000000000000000
2, 1.4166666666666666666666666666666666666666666666667
3, 1.4142156862745098039215686274509803921568627450981
4, 1.4142135623746899106262955788901349101165596221157
5, 1.4142135623730950488016896235025302436149819257762
6, 1.4142135623730950488016887242096980785696718753772
7, 1.4142135623730950488016887242096980785696718753770
8, 1.4142135623730950488016887242096980785696718753770
>
> quit
```

## The Onion Method of Differentiation

- **Onion Method:** Apply the rule that is appropriate for the last operation needed to **evaluate** the expression. Repeat as needed for the ingredients of that expression.
- Examples:

$$f(x) = \frac{\sin x^2}{\sin x^3} \quad = \quad \boxed{\frac{\boxed{\sin \boxed{x^2}}}{\boxed{\sin \boxed{x^3}}}}$$

$$f(x) = \frac{x \sin^2 x^2}{\sqrt{1+x^2}} \quad = \quad \boxed{\frac{\boxed{\boxed{x}\,\boxed{\left(\boxed{\sin \boxed{x^2}}\right)^2}}}{\boxed{\sqrt{\boxed{\boxed{1} + \boxed{x^2}}}}}}$$

- Note: The second function is the subject of problem 19 on hw 5.

- The first example has 5 boxes. Let's label them and state what rules they require:

$$f(x) = \frac{\sin x^2}{\sin x^3} = \boxed{\frac{\boxed{\sin \boxed{x^2}_1}_2}{\boxed{\sin \boxed{x^3}_3}_4}}_5$$

1.

2.

3.

4.

5.

Let's do the actual differentiation:

$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\sin x^2}{\sin x^3} =$

- Here is the same result with the derivatives of the boxed terms indicated by square brackets

$$f(x) = \frac{\sin x^2}{\sin x^3} = \boxed{\frac{\boxed{\sin \boxed{x^2}_1}_2}{\boxed{\sin \boxed{x^3}_3}_4}}_5$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x^2}{\sin x^3} = \left[\frac{\left[\cos x^2\,[2x]_1\right]_2 \sin x^3 - \sin x^2 \left[\cos x^3\,\left[3x^2\right]_3\right]_4}{\sin^2 x^3}\right]_5$$

- **Exercise:** do the same with example 2:

$$f(x) = \frac{x\sin^2 x^2}{\sqrt{1+x^2}} = \boxed{\frac{\boxed{\boxed{x}_1 \; \boxed{\left(\boxed{\sin \boxed{x^2}_2}_3\right)^2}_4}_5}{\boxed{\sqrt{\boxed{\boxed{1}_6 + \boxed{x^2}_7}_8}}_9}}_{10}$$

1.
2.
3.
4.
5.
6.
7.
8.
9.
10.

Now compute the derivative, and mark it with square brackets if you like.

- Of course, you don't want to have to draw boxes and label them.
- Instead handle the onion layers mentally.
- Problem 64, page 124. Find the equation of the tangent line to

  $$y = (x^2+1)^3(x^4+1)^2$$

  at $(1, 32)$.

- Problem 81, page 125. Suppose $f(0) = 0$ and $f'(0) = 2$. What is

$$\frac{\mathrm{d}}{\mathrm{d}x} f(f(f(f(x))))$$

at $x = 0$?

- Use the product rule to derive the power rule for positive integers by induction.