Math 1210-23, Notes of 1/31/24

Announcements

- I get quite a few requests to accept late lab submissions.
- One member of your group will enter one lab sheet for your whole group.
- Make sure your full name and your UID is on that sheet.
- Lab submissions are due by 3:00pm on Thursdays.
- If you participated in the lab and your score does not show up on Canvas within a week or so, first talk to your LA. They know what happened in your lab, I don't. They also have the ability to enter scores into Canvas.

Math 1210 Exam 1 Review

- Our first exam will take place Friday, in our regular classroom. It will cover Chapter 1, Limits.
- Derivatives will be the subject of Exam 2, March 1.
- Several people asked if prerequisites will be on the exam. Not directly. But of course, computing a limit may require that you use some prerequisites. The whole class is based on the assumption that you understnad College Algebra and Trigonometry.
- Since this is our first exam, for your information, here is a copy of the front page. Read it closely so you are familiar with the instructions before the exam.
- The actual exam is printed in a smaller font which accounts for some idiosyncrasies in these notes.

Math 1210-23 Spring 2024 Exam 1February 2, 2024

Clearly write **your name** in this box:

Clearly write **your UID** in this box:

Instructions

1. This exam will be processed by **Gradescope**. This means that grading will be more informative for you, and more consistent. But you should make some extra effort to make gradescope effective. Please use a writing utensil that draws a dark curve (not a hard pencil, for example). For each problem, please enter your final answer, **and nothing else**, in the box(es) provided with the question. Everywhere, but particularly in those boxes, make an effort to **write clearly and avoid crossing out things** if you can. Write all work in the remaining space provided with the problem. If you write a note or comment for me in a random spot I may miss it. Therefore, at the end of the exam there is an extra large box that you can use for additional work or for notes to me. Most people will leave that box blank. However, I will have Gradescope treat that box as an ungraded exam question.

That way, gradescope will make me look at the box if it is not left blank.

2. This exam is **closed books and notes, no electronic devices, no scratch paper**.
3. Please note: **To avoid disruption and distraction I won't be able to answer questions during the exam. If you believe there is a mistake in one of the problems write down an appropriate note and if you are right you will receive generous credit.**
4. The questions on this test are deliberately simple. Also, you have seen all of these questions before, in class, or on the home works! You should not be rushed and have time to answer all questions carefully and check your answers. **Accuracy is more important than speed.** Don't get stuck on any one problem. If you can't answer a question immediately go on and return to that question only after you have answered the others.
5. **Since the problems are taken from our past class work and the home work assignments, as promised, you may remember the problem and its answer. In that case it is not enough to simply enter the answer. You need to give some reason or calculation. If you believe the answer is truly obvious then state so!**
6. Simplify any algebraic expressions and reduce any fractions.

7. You don't need to approximate mathematical expressions with decimal expressions.

8. If you are done before the allotted time is up I recommend that you stay and use the remaining time to check your answers.

9. All questions have equal weight.

Math 1210-23

Notes of 1/31/24

- The exam will have 8 questions covering chapters 1 of our textbook.
- All questions are taken, verbatim or with small modifications, from home works 1–3 or past class work.
- If you happen to remember the answer it's not enough to state it, you need to show how to obtain it.
- The only exception is the last question which has a bunch of T/F items. There, you don't need to give reasons fro your answer.

Review of Chapter 1

These notes are neither complete nor self contained. Rather, they should trigger your recollection of our discussions during the first three weeks of the semester. If they don't review the relevant part of your notes or the textbook!

- The key subjects in Chapter 1 are **limits** and **continuity**. The concept of limits is needed to define derivatives, which form the heart of Calculus.

The most important kind of limit is the quotient of two expressions where both numerator and denominator approach zero. We use that kind of limit to define derivatives, and being able to define derivatives is the reason we study limits.

The expression

$$\frac{0}{0} \tag{1}$$

is undefined and does not make sense. However, the limit

$$\lim_{h \longrightarrow 0} \frac{N(h)}{D(h)} \tag{2}$$

where $N(0) = D(0) = 0$ may make perfect sense and define a derivative, depending on the precise circumstances.

- Intuitively,

$$\lim_{x \longrightarrow c} f(x) = L \tag{3}$$

  means that $f(x)$ gets arbitrarily close to L as x goes sufficiently close to c.

- We say that *the limit of $f(x)$ as x approaches c equals L.*

- We can make $f(x)$ as close to L as we wish, all we have to do is pick x as close to c as we have to.

- Think of it as a competition. You challenge me to get $f(x)$ within a certain distance from L. I get to tell you how closely you have to pick x to c.

- I have to tell you how to do this in general. That's why we use variables instead of specific numbers.

- **Formally**,

$$\lim_{x \longrightarrow c} f(x) = L \tag{4}$$

if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $|x - c| < \delta$.

Very major Point: It does not matter for the limit what happens to $f(x)$ when $x = c$. $f(c)$ may be undefined, or its value may be distinct from the limit, without changing the truth of the equation (4).

Major Technique: Don't worry about $f(c)$, replace $f(x)$ with an equivalent expression where whatever difficulty we have at $x = c$ has disappeared.

- Example. Compute

$$\lim_{x \longrightarrow \frac{\pi}{4}} \frac{\sin^2 x - \cos^2 x}{\sin x - \cos x} = \lim_{x \to \frac{\pi}{4}} \frac{\cancel{(\sin x - \cos x)}(\sin x + \cos x)}{\cancel{\sin x - \cos x}}$$

$$= \sqrt{2}$$

Helpful hint:

$$\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$$

- Example: compute

$$\lim_{h \longrightarrow 0} \frac{(x+h)^2 - x^2}{h}. \tag{5}$$

(This happens to be the derivative of x^2, but the point is illustrating how to compute the limit, not interpreting it.)

$$\frac{(x+h)^2 - x^2}{h} = \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h}$$

$$= 2x + h \to 2x$$

- Of course, sometimes there is no trouble at $x = c$, we can just evaluate.
- Example: compute

$$\lim_{x \longrightarrow 0} \frac{x}{x^2 + 1} \quad = 0 \tag{6}$$

Main Limit Theorem

- See textbook, page 68.
- Let n be a positive integer, k a constant, and f and g functions that have limits at c. Then:

1. $\lim_{x \longrightarrow c} k = k$.

2. $\lim_{x \longrightarrow c} x = c$.

3. $\lim_{x \longrightarrow c} kf(x) = k \lim_{x \longrightarrow c} f(x)$.

4. $\lim_{x \longrightarrow c} \big(f(x) + g(x)\big) = \lim_{x \longrightarrow c} f(x) + \lim_{x \longrightarrow c} g(x)$.

5. $\lim_{x \longrightarrow c} \big(f(x) - g(x)\big) = \lim_{x \longrightarrow c} f(x) - \lim_{x \longrightarrow c} g(x)$.

6. $\lim_{x \longrightarrow c} \big(f(x) \cdot g(x)\big) = \Big(\lim_{x \longrightarrow c} f(x)\Big) \cdot \Big(\lim_{x \longrightarrow c} g(x)\Big)$.

7. $\lim_{x \longrightarrow c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \longrightarrow c} f(x)}{\lim_{x \longrightarrow c} g(x)}$ provided $\lim_{x \longrightarrow c} g(x) \neq 0$.

8. $\lim_{x \longrightarrow c} \big(f(x)\big)^n = \Big(\lim_{x \longrightarrow c} f(x)\Big)^n$.

9. $\lim_{x \longrightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \longrightarrow c} f(x)}$ provided $f(x) \geq 0$ when n is even.

The Squeeze Theorem

Suppose f, g, and h are functions such that

$$f(x) \leq g(x) \leq h(x) \tag{7}$$

for all x near c except possibly *at* $x = c$. Also assume that

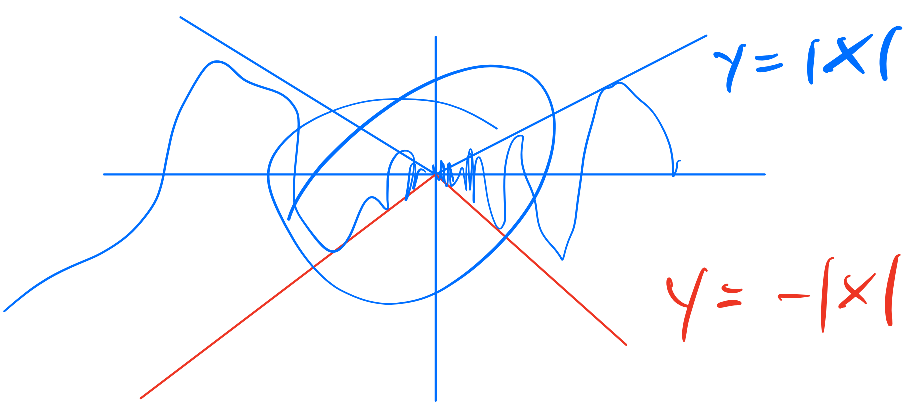
$$\lim_{x \longrightarrow c} f(x) = \lim_{x \longrightarrow c} h(x) = L \tag{8}$$

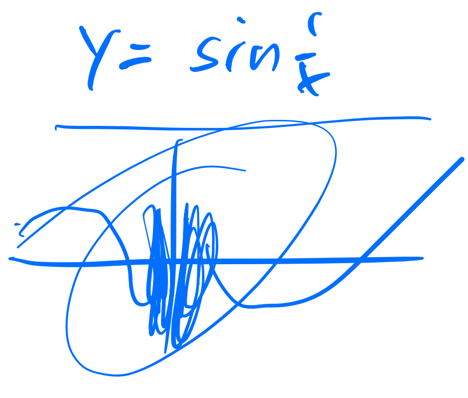
Then

$$\lim_{x \longrightarrow c} g(x) = L \tag{9}$$

- For example, show that

$$\lim_{x \longrightarrow 0} x \sin \frac{1}{x} = 0.$$





One Sided Limits

- These work exactly like limits, except that we assume either that $x > c$, or $x < c$. We denote such limits as

$$\lim_{x \longrightarrow c^-} f(x) = \lim_{\substack{x \longrightarrow c \\ x < c}} f(x) \tag{10}$$

and

$$\lim_{x \longrightarrow c^+} f(x) = \lim_{\substack{x \longrightarrow c \\ x > c}} f(x) \tag{11}$$

- Example: the greatest integer function:

$$[[x]] = \text{the greatest integer} \leq x. \tag{12}$$

$$\lim_{x \longrightarrow 1^+} [[x]] = 1 \qquad \text{and} \qquad \lim_{x \longrightarrow 1^-} [[x]] = 0 \tag{13}$$

- **Limits at infinity:** For example

$$\lim_{x \longrightarrow \infty} \frac{1}{x^2 + 1} = 0. \tag{14}$$

- **Infinite Limits:** For example,

$$\lim_{x \longrightarrow 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \longrightarrow 0^-} \frac{1}{x} = -\infty. \tag{15}$$

- If a limit is infinite it does not exist. For example

$$\lim_{x \longrightarrow 0} \frac{1}{x^2} = \infty \tag{16}$$

and

$$\lim_{x \longrightarrow 0} \frac{1}{x^2} \qquad \text{does not exist} \tag{17}$$

are both true statements.

- We discussed a few trigonometric limits, specifically

$$\lim_{x \longrightarrow 0} \frac{x}{\sin x} = \lim_{x \longrightarrow 0} \frac{\sin x}{x} = 1 \tag{18}$$

and

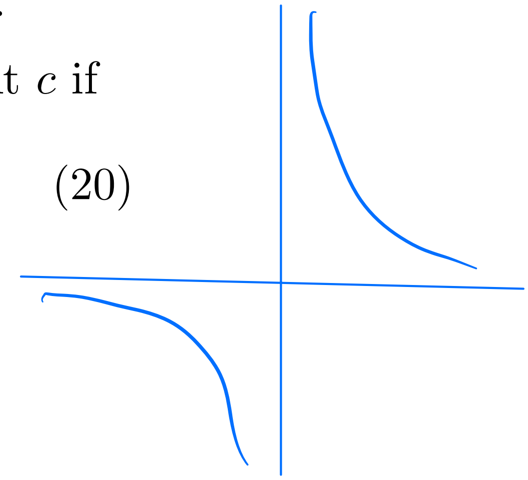
$$\lim_{x \longrightarrow 0} \frac{1 - \cos x}{x} = 0. \tag{19}$$

Continuity

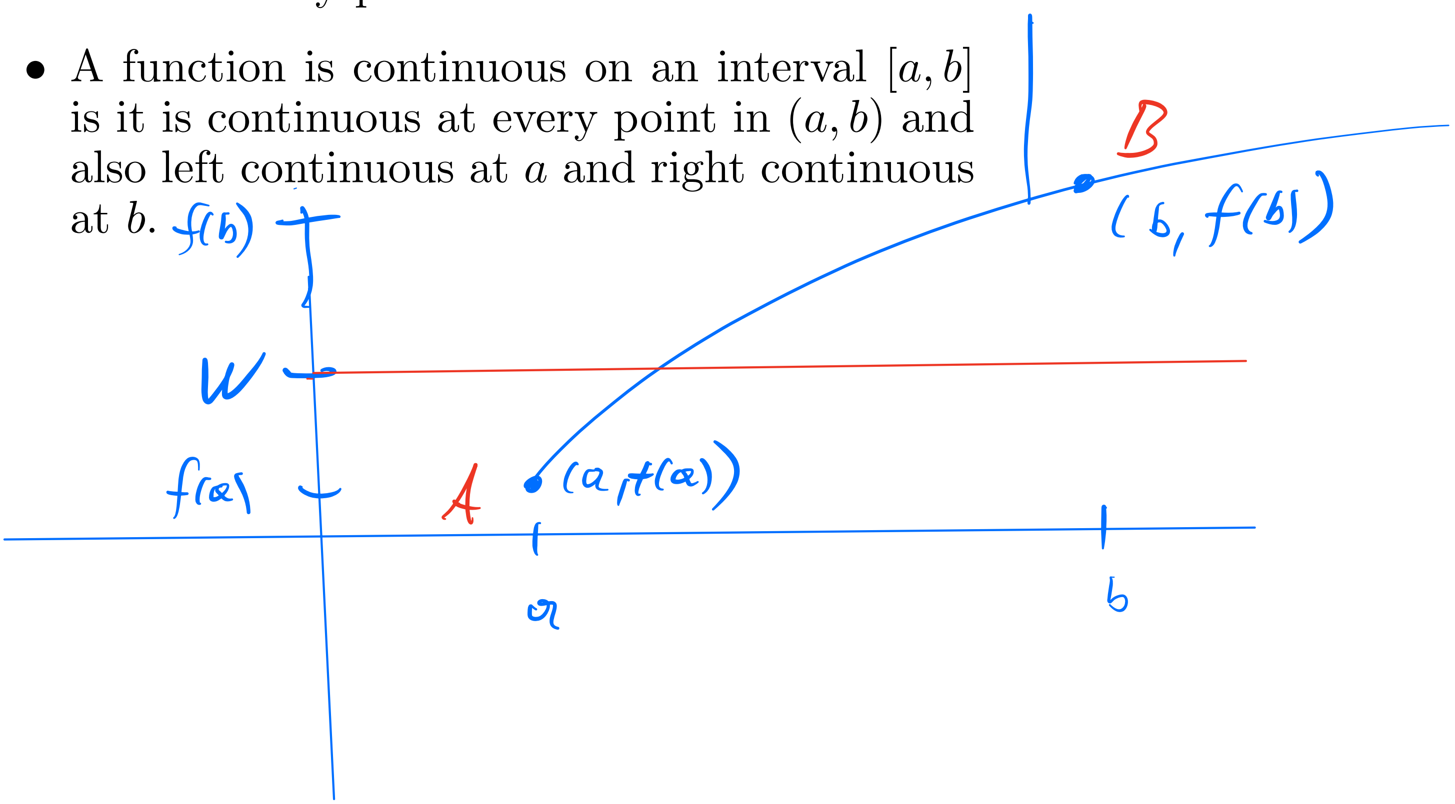
- Intuitively, a function is continuous if we can draw its graph without lifting the pen.
- The function f is continuous at a point c if

$$\lim_{x \longrightarrow c} f(x) = f(c). \tag{20}$$

  This definition has three ingredients:
  - The limit on the left exists
  - c is in the domain of f
  - The equality holds.



- Similarly to left and right sided limits we can define left and right sided continuity.
- A function is just **continuous** if it is continuous at every point in its domain.
- A function is continuous on an interval $[a, b]$ is it is continuous at every point in (a, b) and also left continuous at a and right continuous at b.



The intermediate value theorem

- Page 87, textbook:

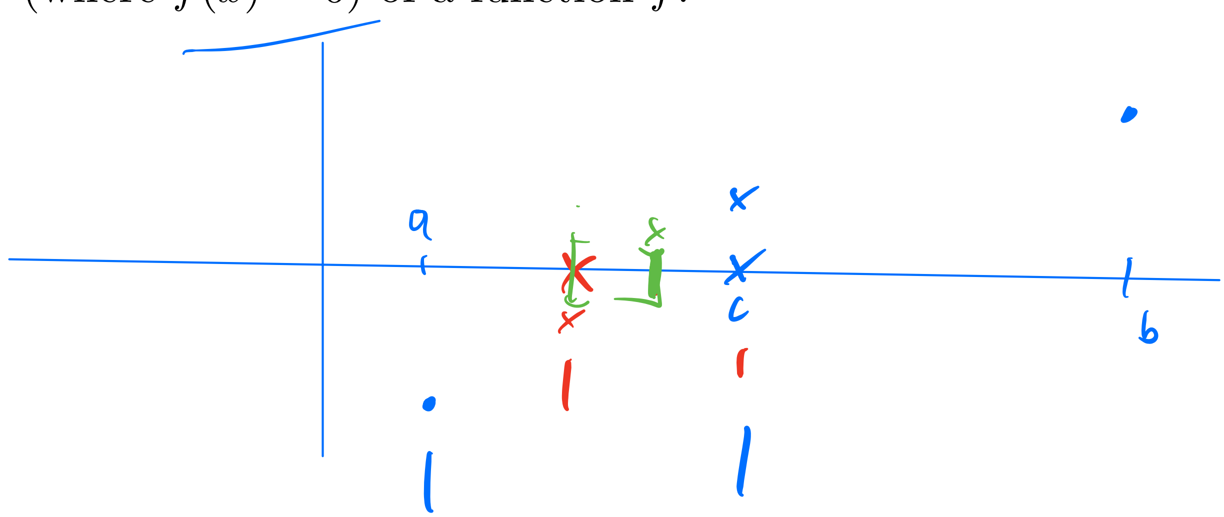
  Let f be defined on $[a, b]$. Suppose

$$f(a) \leq W \leq f(b). \tag{21}$$

Then there is at least one number c such that

$$a \leq c \leq b \quad \text{and} \quad f(c) = W \tag{22}$$

- Polynomials are continuous everywhere.
- Rational functions are continuous everywhere except where the denominator is zero.
- Sums, differences, products, and compositions of continuous functions are continuous.
- Quotients of continuous functions are continuous unless the denominator is zero.
- An application of continuity is the bisection method for the approximation of the zeros (where $f(x) = 0$) of a function f.



$$\sin(t^\circ) = \sin\left(\frac{t \cdot \pi}{180}\ \text{rad}\right)$$

$$\lim \frac{\sin\left(\frac{t\pi}{180}\right)}{t\,\frac{\pi}{180}} \cdot \frac{\pi}{180}$$