Announcements

- tomorrow, Wednesday: review of Chapter 1.
- Friday, 2/2/24: Exam 1 on Chapter 1, Limits.
- Derivatives will be the contents of Exam 2, 3/1/24
- I usually proctor exams myself, but this particular exam will be run by Andy. I am also unable to hold office hours on Friday, ~~April~~ 2, 2024
  February

Taking Stock

- Suppose f and g are functions of x, k is constant, and n is a positive integer.
- So far, we have the following **differentiation rules:**

$(kf)'$	$=$	kf'	**Constant Multiple Rule**
$(f+g)'$	$=$	$f'+g'$	**Sum Rule**
$(x^n)'$	$=$	nx^{n-1}	**Power Rule**
$(fg)'$	$=$	$f'g+fg'$	**Product Rule**
$\left(\frac{f}{g}\right)'$	$=$	$\frac{f'g-fg'}{g^2}$	**Quotient Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\sin x$	$=$	$\cos x$	**Sine Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\cos x$	$=$	$-\sin x$	**Cosine Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x}$	$=$	$-\frac{1}{x^2}$	

- The first and second property together are equivalent to saying that

Differentiation is Linear

2.5 The Chain Rule

- Recall function composition

$$f \circ g(x) = f\big(g(x)\big).$$

- The Chain Rule can be written as

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) = f'\big(g(x)\big) g'(x).}$$

- In other words, **the derivative of the composition is the product of the derivatives.**

Examples

$\frac{\mathrm{d}}{\mathrm{d}x}(x^2+1)^{100} = f'(g(x))\, g'(x) = 100(x^2+1)^{99} \cdot 2x$

$f(x) = x^{100}$

$g(x) = x^2+1$

$= 200x(x^2+1)^{99}$

$$\frac{d}{dx}\sin^2 x = \ = f'(g(x))\, g'(x) = 2\sin x \cdot \cos x$$

$f(x) = x^2 \qquad f'(x) = 2x$

$g(x) = \sin x \qquad g'(x) = \cos x$

$$\frac{d}{dx}\sin^2 x = \frac{d}{dx}\sin x \sin x = \cos x \sin x + \sin x \cos x$$

$$= 2\sin x \cos x$$

$$\frac{d}{dx}\sin x^2 = \left(\cos x^2\right) 2x = 2x \cos x^2$$

$f(x) = \sin x \qquad f'(x) = \cos x$

$g(x) = x^2 \qquad g'(x) = 2x$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x^2}{\sin x} = \frac{(\cos x^2)\,2x\,\sin x - \sin x^2\,\cos x}{\sin^2 x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin^2 x}{\sin x} = \frac{2\sin x\cos x\sin x - \sin^2 x\cos x}{\sin^2 x}$$

$$= 2\cos x - \cos x = \cos x$$

$$=$$

$$\frac{d}{dx}\frac{\sin^2 x}{\sin x} = \frac{d}{dx}\sin x = \cos x$$

- The Power Rule says

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}, \qquad n = 1, 2, 3, \ldots$$

- We also know the chain rule and

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x} = -\frac{1}{x^2}$$

- Query: What is the derivative of

$$f(x) = x^{-n}.$$

$$f(x) = x^{-1} = \left(\frac{1}{x}\right)^n$$

$$\frac{d}{dx}\left(\frac{1}{x}\right)^n = -n\left(\frac{1}{x}\right)^{n-1}\frac{1}{x^2} = -n\frac{1}{x^{n-1+2}}$$

$$= -n\frac{1}{x^{n+1}}$$

$$= -n\,x^{-(n+1)} = -n x^{-n-1}$$

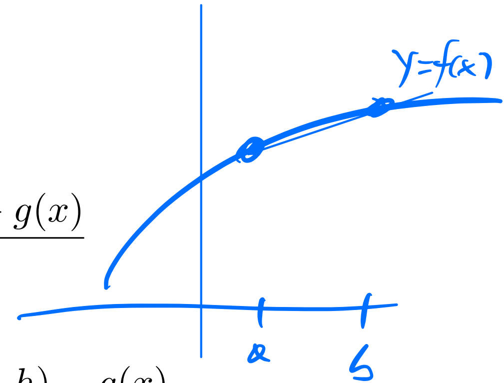
- Can we use the quotient rule?

$$\frac{d}{dx}x^{-n} = \frac{d}{dx}\frac{1}{x^n} = \frac{-nx^{n-1}}{(x^n)^2}$$

$$= -n\frac{x^{n-1}}{x^{2n}} = -n\,x^{n-1}x^{-2n}$$

$$= -n\,x^{n-1-2n} = -n x^{-n-1}$$

- Why does the Chain Rule work?
- This is actually a little tricky, see the textbook.
- But here is a very compelling more casual argument:



$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) &= \lim_{h \longrightarrow 0} \frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{h} \\
&= \lim_{h \longrightarrow 0} \frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{g(x+h) - g(x)} \times \frac{g(x+h) - g(x)}{h} \\
&= \underbrace{\lim_{h \longrightarrow 0} \frac{f\big(\overset{a}{g(x+h)}\big) - f\big(\overset{b}{g(x)}\big)}{g(x+h) - g(x)}}_{= f'(g(x))} \times \underbrace{\lim_{h \longrightarrow 0} \frac{g(x+h) - g(x)}{h}}_{= g'(x)} \\
&= f'\big(g(x)\big) g'(x).
\end{aligned}$$

What can go wrong?

- Here is another more suggestive way, using Leibniz Notation:
- Let
$$u = g(x).$$
- Then, by the chain rule
$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) = f'(u)g'(x) = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$
- The $\mathrm{d}u$ "cancel"
- Leibniz notation is often used for this kind of mental crutch.

More Examples

$\frac{\mathrm{d}}{\mathrm{d}x}\sin(\cos(x)) = -\cos(\cos x)\sin x$

$\frac{d}{dx}\sin x = \cos x \cdot \frac{dx}{dx} = \cos x$

$\frac{\mathrm{d}}{\mathrm{d}x}\sin(\cos x^2) = -\cos(\cos x^2)\sin x^2 \cdot 2x$

$= -2x\cos(\cos x^2) \cdot \sin x^2$

$\frac{\mathrm{d}}{\mathrm{d}x}\sin(\cos^2 x) = -\cos(\cos^2 x)\, 2\cos x \sin x$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3t-2}{t+5}\right)^2 = 2\left(\frac{3t-2}{t+5}\right)\frac{3(t+5)-(3t-2)}{(t+5)^2}$$

simplify . . .

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s^2-9}{s+4}\right)^2 = 2\left(\frac{s^2-9}{s+4}\right)\frac{2s(s+4)-(s^2-9)}{(s+4)^2}$$

simplify

- Recall: a function is **even** if

$$f(x) = f(-x)$$

for all x in the domain, and it is **odd** if

$$f(x) = -f(-x)$$

- Show that the derivative of an even function is odd.

$$f(x) = f(-x)$$

$$f'(x) = -f'(-x)$$

- Similarly, the derivative of an odd function is even.

- More Examples:

  $\frac{\mathrm{d}}{\mathrm{d}x}\frac{(x^2+x)^4-\sin x}{2-\sin x} =$

  $\frac{\mathrm{d}}{\mathrm{d}x}\sin\frac{1}{x} - \frac{1}{\sin x} =$