Announcements

- tomorrow, Wednesday: review of Chapter 1.
- Friday, 2/2/24: Exam 1 on Chapter 1, Limits.
- Derivatives will be the contents of Exam 2, 3/1/24
- I usually proctor exams myself, but this particular exam will be run by Andy. I am also unable to hold office hours on Friday, April 2, 2024
Taking Stock

- Suppose \( f \) and \( g \) are functions of \( x \), \( k \) is constant, and \( n \) is a positive integer.
- So far, we have the following differentiation rules:

\[
\begin{align*}
(kf)' &= kf' & \text{Constant Multiple Rule} \\
(f + g)' &= f' + g' & \text{Sum Rule} \\
(x^n)' &= nx^{n-1} & \text{Power Rule} \\
(fg)' &= f'g + fg' & \text{Product Rule} \\
\left(\frac{f}{g}\right)' &= \frac{f'g - fg'}{g^2} & \text{Quotient Rule} \\
\frac{d}{dx} \sin x &= \cos x & \text{Sine Rule} \\
\frac{d}{dx} \cos x &= -\sin x & \text{Cosine Rule} \\
\frac{d}{dx} \frac{1}{x} &= -\frac{1}{x^2}
\end{align*}
\]

- The first and second property together are equivalent to saying that

Differentiation is Linear
2.5 The Chain Rule

• Recall function composition

\[ f \circ g(x) = f(g(x)). \]

• The Chain Rule can be written as

\[ \frac{d}{dx} f(g(x)) = f'(g(x))g'(x). \]

• In other words, the derivative of the composition is the product of the derivatives.

Examples

\[
\frac{d}{dx} (x^2 + 1)^{100} = f'(g(x)) g'(x) = 100 (x^2 + 1)^9 2x \\
f(x) = x^{100} \\
g(x) = x^2 + 1
\]

\[ = 200x(x^2 + 1)^9 \]
\[
\frac{d}{dx} \sin^2 x = \left( g(x^2) \right) g'(x) = 2 \sin x \cdot \cos x
\]
\[
f(x) = \sin x \quad f'(x) = \cos x
\]
\[
g(x) = x^2 \quad g'(x) = 2x
\]
\[
\frac{d}{dx} \sin^2 x = \frac{d}{dx} \sin x \cdot \sin x = \cos x \cdot \sin x
\]
\[
+ \sin x \cdot \cos x
\]
\[
= 2 \sin x \cdot \cos x
\]

\[
\frac{d}{dx} \sin x^2 = \left( \cos x^2 \right) 2x = 2x \cdot \cos x^2
\]
\[
f(x) = \sin x \quad f'(x) = \cos x
\]
\[
g(x) = x^2 \quad g'(x) = 2x
\]
\[
\frac{d}{dx} \sin^2 x = \frac{(\cos x^2) 2x \sin x - \sin x^2 \cos x}{\sin^2 x}
\]

\[
\frac{d}{dx} \sin^2 x = \frac{2 \sin x \cos x \sin x - \sin^2 x \cos x}{\sin^2 x}
\]

\[
= 2 \cos x - \cos x = \cos x
\]

\[
\frac{d}{dx} \frac{\sin^2 x}{\sin x} = \frac{d}{dx} \sin x = \cos x
\]
• The Power Rule says

\[
\frac{d}{dx} x^n = nx^{n-1}, \quad n = 1, 2, 3, \ldots
\]

• We also know the chain rule and

\[
\frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2}
\]

• Query: What is the derivative of

\[
f(x) = x^{-n}.
\]

\[
f'(x) = x^{-1} = \left(\frac{1}{x}\right)^n
\]

\[
\frac{d}{dx} \left(\frac{1}{x}\right)^n = -n \left(\frac{1}{x}\right)^{n-1} \frac{1}{x^2} = -n \frac{1}{x^{n-1} + 2}
\]

\[
= -n \frac{1}{x^{n+1}}
\]

\[
= -n x^{-(n+1)} = -n x^{-n-1}
\]

• Can we use the quotient rule?

\[
\frac{d}{dx} x^{-n} = \frac{d}{dx} \frac{1}{x^n} = -n x^{n-1} \frac{1}{(x^n)^2}
\]

\[
= -n \frac{x^{n-1}}{x^{2n}} = -n x^{n-1} \frac{x}{x^{2n}} = -n x^{n-1} \frac{x}{x^{2n}} = -n x^{n-1} \frac{x}{x^{2n}} = -n x^{n-1} \frac{x}{x^{2n}} = -n x
\]
• Why does the Chain Rule work?
• This is actually a little tricky, see the textbook.
• But here is a very compelling more casual argument:

\[
\begin{align*}
\frac{d}{dx} f(g(x)) &= \lim_{h \to 0} \frac{f(g(x + h)) - f(g(x))}{h} \\
&= \lim_{h \to 0} \frac{f(g(x + h)) - f(g(x))}{g(x + h) - g(x)} \times \frac{g(x + h) - g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(a) - f(b)}{a - b} \times \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} \\
&= f'(g(x)) \times g'(x).
\end{align*}
\]

What can go wrong?
Here is another more suggestive way, using Leibniz Notation:

Let \( u = g(x) \).

Then, by the chain rule,

\[
\frac{df}{dx} = \frac{d}{dx} f(g(x)) = f'(u)g'(x) = \frac{df}{du} \frac{du}{dx}.
\]

The \( du \) “cancel”

Leibniz notation is often used for this kind of mental crutch.
More Examples

\[ \frac{d}{dx} \sin(\cos(x)) = -\cos(\cos(x)) \sin x \]

\[ \frac{d}{dx} \sin x = \cos x \cdot \frac{dx}{dx} = \cos x \]

\[ \frac{d}{dx} \sin(\cos(x^2)) = -\cos(\cos(x^2)) \sin x^2 \cdot 2x \]

\[ = -2x \cos(\cos(x^2)) \cdot \sin x^2 \]

\[ \frac{d}{dx} \sin(\cos^2 x) = -\cos(\cos^2 x) \cdot 2 \cos x \cdot \sin x \]
\[ \frac{d}{dt} \left( \frac{3t-2}{t+5} \right)^2 = 2 \left( \frac{3t-2}{t+5} \right) \frac{3(t+5)-(3t-2)}{(t+5)^2} \]

simplify...

\[ \frac{d}{ds} \left( \frac{s^2-9}{s+4} \right)^2 = 2 \left( \frac{s^2-9}{s+4} \right) \frac{2s(s+4)-(s^2-9)}{(s+4)^2} \]

simplify
• Recall: a function is **even** if

\[ f(x) = f(-x) \]

for all \( x \) in the domain, and it is **odd** if

\[ f(x) = -f(-x) \]

• Show that the derivative of an even function is odd.

\[ f'(x) = f'(-x) \]

\[ f'(x) = -f'(-x) \]

• Similarly, the derivative of an odd function is even.
• More Examples:
\[
\frac{d}{dx} \left( \frac{(x^2+x)^4 - \sin x}{2 - \sin x} \right) =
\]

\[
\frac{d}{dx} \sin \frac{1}{x} - \frac{1}{\sin x} =
\]