Announcements

- tomorrow, Wednesday: review of Chapter 1.
- Friday, 2/2/24: Exam 1 on Chapter 1, Limits.
- Derivatives will be the contents of Exam 2, 3/1/24
- I usually proctor exams myself, but this particular exam will be run by Andy. I am also unable to hold office hours on Friday, April 2, 2024

Taking Stock

- Suppose f and g are functions of x, k is constant, and n is a positive integer.
- So far, we have the following **differentiation rules:**

$(kf)'$	$=$	kf'	**Constant Multiple Rule**
$(f+g)'$	$=$	$f'+g'$	**Sum Rule**
$(x^n)'$	$=$	nx^{n-1}	**Power Rule**
$(fg)'$	$=$	$f'g+fg'$	**Product Rule**
$\left(\frac{f}{g}\right)'$	$=$	$\frac{f'g-fg'}{g^2}$	**Quotient Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\sin x$	$=$	$\cos x$	**Sine Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\cos x$	$=$	$-\sin x$	**Cosine Rule**
$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x}$	$=$	$-\frac{1}{x^2}$	

- The first and second property together are equivalent to saying that

Differentiation is Linear

2.5 The Chain Rule

- Recall function composition

$$f \circ g(x) = f\big(g(x)\big).$$

- The Chain Rule can be written as

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) = f'\big(g(x)\big)g'(x).}$$

- In other words, **the derivative of the composition is the product of the derivatives.**

Examples

$\frac{\mathrm{d}}{\mathrm{d}x}(x^2+1)^{100} =$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^2 x =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x^2 =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x^2}{\sin x} =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin^2 x}{\sin x} =$$

- The Power Rule says

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}, \qquad n = 1, 2, 3, \ldots$$

- We also know the chain rule and

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x} = -\frac{1}{x^2}$$

- Query: What is the derivative of

$$f(x) = x^{-n}.$$

- Can we use the quotient rule?

- Why does the Chain Rule work?
- This is actually a little tricky, see the textbook.
- But here is a very compelling more casual argument:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) &= \lim_{h \longrightarrow 0} \frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{h} \\
&= \lim_{h \longrightarrow 0} \frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{g(x+h) - g(x)} \times \frac{g(x+h) - g(x)}{h} \\
&= \underbrace{\lim_{h \longrightarrow 0} \frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{g(x+h) - g(x)}}_{= f'\left(g(x)\right)} \times \underbrace{\lim_{h \longrightarrow 0} \frac{g(x+h) - g(x)}{h}}_{= g'(x)} \\
&= f'\big(g(x)\big) g'(x).
\end{aligned}$$

What can go wrong?

- Here is another more suggestive way, using Leibniz Notation:
- Let
$$u = g(x).$$
- Then, by the chain rule
$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) = f'(u)g'(x) = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$
- The $\mathrm{d}u$ "cancel"
- Leibniz notation is often used for this kind of mental crutch.

More Examples

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(\cos(x)) =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(\cos x^2) =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(\cos^2 x) =$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3t-2}{t+5}\right)^2 =$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s^2-9}{s+4}\right)^2 =$$

- Recall: a function is **even** if

$$f(x) = f(-x)$$

for all x in the domain, and it is **odd** if

$$f(x) = -f(-x)$$

- Show that the derivative of an even function is odd.

- Similarly, the derivative of an odd function is even.

- More Examples:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{(x^2+x)^4-\sin x}{2-\sin x} =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin\frac{1}{x} - \frac{1}{\sin x} =$$