Math 1210–23 Notes of 1/29/24

- So far we know these differentiation rules:
- **Power Rule:**

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}}$$

where n is a positive integer.

- **Sum Rule:**

$$\boxed{(f+g)' = f' + g'}$$

- **Constant Multiple Rule:**

$$\boxed{(kf)' = k(f')}$$

where k is a constant.

- With the above we can differentiate any polynomial!
- We also have these two miscellaneous rules:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x} = \frac{-1}{x^2}.$$

- But we need more rules!

More Differentiation Rules

The Product Rule

- The derivative of the product does not in general equal the product of the derivatives!
- Instead:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} f(x)g(x) = f'(x)g(x) + f(x)g'(x)},$$

or, more briefly

$$\boxed{(uv)' = u'v + uv'}$$

- One way to remember the product rule is to note that each of the two term can be obtained by considering one of the factors constant.

- Example: Compute

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2+2)(x^3+1)$$

in two different ways:

$$\frac{d}{dx}(x^2+2)(x^3+1) = 2x(x^3+1) + (x^2+2)3x^2$$
$$= 2x^4+2x + 3x^4 + 6x^2$$
$$= 5x^4+6x^2+2x$$
$$= \frac{d}{dx}(x^5+2x^3+x^2+2)$$
$$= 5x^4+6x^2+2x$$

- Example: Compute

$$\frac{\mathrm{d}}{\mathrm{d}x}(x+\sqrt{x})(x^2+x) =$$

$$= \left(1+\frac{1}{2\sqrt{x}}\right)(x^2+x) + (x+\sqrt{x})(2x+1)$$
$$= x^2+x+\frac{x^{3/2}}{2}+\frac{1}{2}\sqrt{x} + 2x^2 + x + 2x^{3/2} + \sqrt{x}$$
$$= 3x^2+2x+\frac{5}{2}x^{3/2}+\frac{3}{2}\sqrt{x}$$

- Why does the product rule work?
- We go back to the definition of the derivative and apply the Main Limit Theorem (the limit of the sum is the sum of the limits, etc.).

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x)g(x) = \lim_{h \longrightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$\begin{aligned}
&= \lim_{h \longrightarrow 0} \frac{1}{h} \Big(f(x+h)g(x+h) - f(x)g(x+h) \\
&\qquad + f(x)g(x+h) - f(x)g(x) \Big) \\
&= \lim_{h \longrightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} \\
&\qquad + \lim_{h \longrightarrow 0} \frac{f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h} \times \lim_{h \longrightarrow 0} g(x+h) \\
&\qquad + f(x) \lim_{h \longrightarrow 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}$$

- We'll see much more of the product rule when we learn how to differentiate functions other than polynomials.

The Quotient Rule

- The quotient rule is:

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$$

- Examples:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1+x^2} = \frac{0\cdot(1+x^2) - 1\cdot 2x}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x-1}{1+x^2} = \frac{1+x^2 - (x-1)2x}{(1+x^2)^2} = \frac{1-x^2+2x}{(1+x^2)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x+3}{x+4} = \frac{x+4-(x+3)}{(x+4)^2} = \frac{1}{(x+4)^2}$$

$n > 0$ integer

$$\frac{\mathrm{d}}{\mathrm{d}x}x^{-n} = \frac{d}{dx}\frac{1}{x^n} = \frac{-1\cdot nx^{n-1}}{(x^n)^2} = \frac{-nx^{n-1}}{x^{2n}}$$

$$= -nx^{n-1}\cdot x^{-2n} = -nx^{n-1-2n}$$

$$\underline{= -nx^{-n-1}}$$

$$\frac{d}{dx}x^{-4} = -4x^{-5}$$

- Why does the quotient rule work?
- Again, we do algebra, and apply the main limit theorem.

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \lim_{h\longrightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\longrightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}$$

$$= \lim_{h\longrightarrow 0} \frac{1}{g(x+h)g(x)}$$

$$\times \lim_{h\longrightarrow 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h}$$

$$= \lim_{h\longrightarrow 0} \frac{1}{g(x+h)g(x)}$$

$$\times \lim_{h\longrightarrow 0} \frac{f(x+h)g(x) - f(x)g(x)}{h}$$

$$\times \lim_{h\longrightarrow 0} \frac{f(x)g(x) - f(x)g(x+h)}{h}$$

$$= \lim_{h\longrightarrow 0} \frac{1}{g(x+h)g(x)}$$

$$\times \lim_{h\longrightarrow 0} \frac{\big(f(x+h) - f(x)\big)g(x)}{h}$$

$$\times \lim_{h\longrightarrow 0} \frac{f(x)\big(g(x) - g(x+h)\big)}{h}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Derivatives of Trig Functions

- Can you guess the derivative of the sin function?

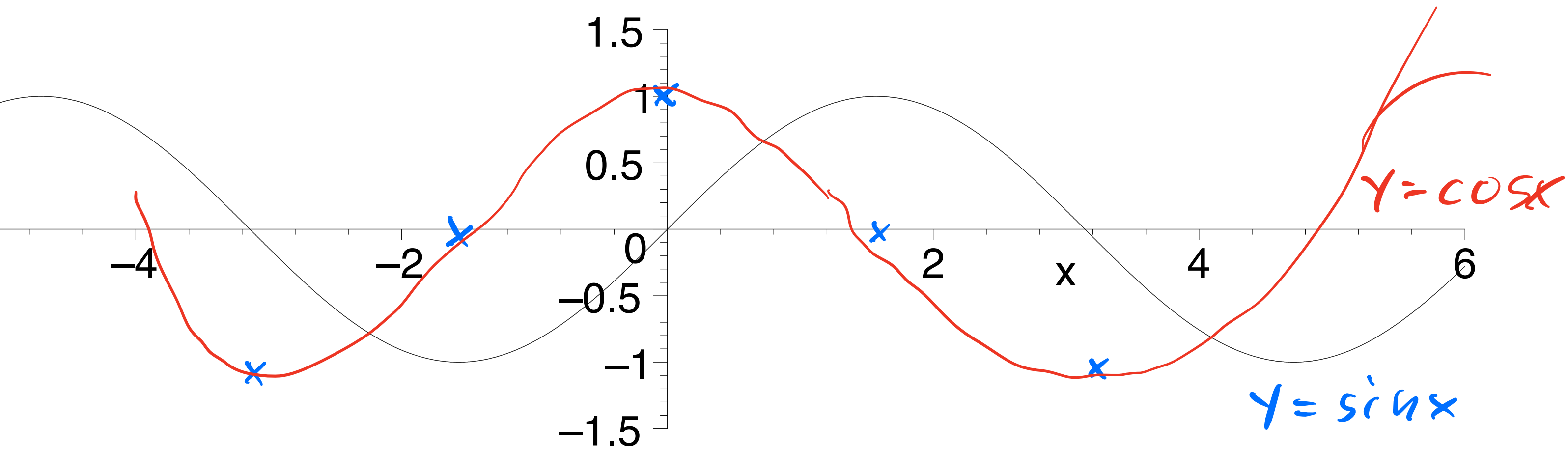

Figure 1. The Sine Function.

- Recall the graphs of sine and cosine:

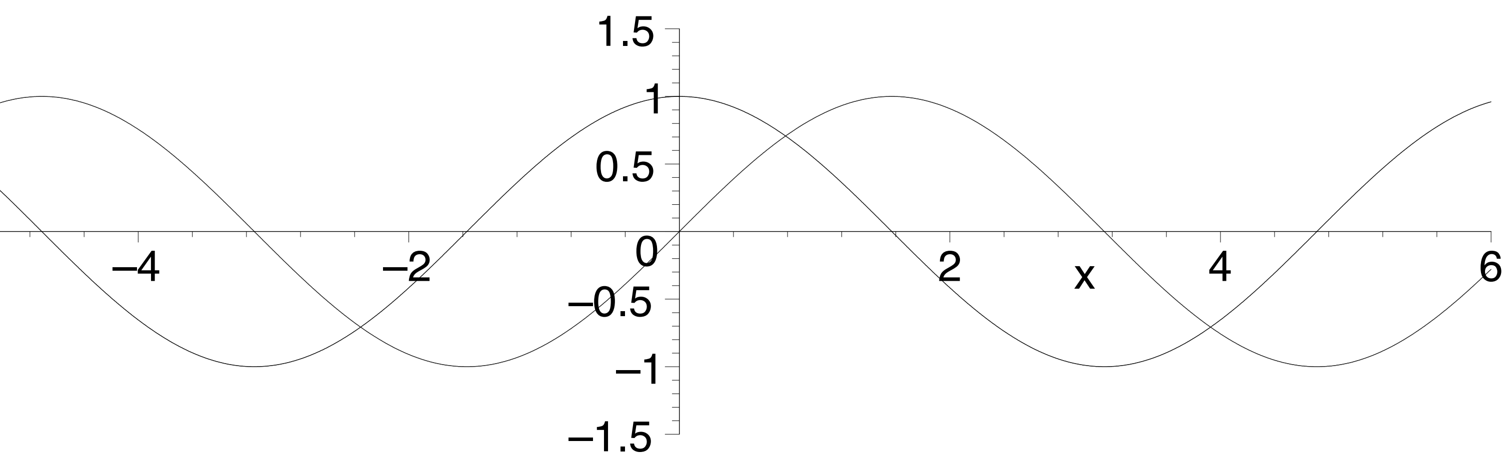

Figure 2. Graphs of sine and cosine.

- Yes indeed!

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x.$$

- Need trig identity:

$$\sin(x+h) = \sin x \cos h + \cos x \sin h.$$

- Also remember these limits:

$$\lim_{h\longrightarrow 0} \frac{\cos x - 1}{x} = 0 \quad \text{and} \quad \lim_{h\longrightarrow 0} \frac{\sin x}{x} = 1.$$

- Again, we go back to the definition and use the main limit theorem:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \lim_{h\longrightarrow 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$\begin{aligned}
&= \lim_{h\longrightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\longrightarrow 0} \sin x \times \frac{\cos h - 1}{h} + \lim_{h\longrightarrow 0} \cos x \frac{\sin h}{h} \\
&= \sin x \underbrace{\lim_{h\longrightarrow 0} \frac{\cosh - 1}{h}}_{=0} + \cos x \underbrace{\lim_{h\longrightarrow 0} \frac{\sin h}{h}}_{=1} \\
&= \cos x.
\end{aligned}$$

- Proceeding similarly (Textbook, page 115) we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x.$$

- More examples:

$$\frac{\mathrm{d}}{\mathrm{d}x}x^2\sin x = 2x\sin x + x^2\cos x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^2 x = \frac{d}{dx}(\sin x)(\sin x) = (\cos x)(\sin x) + (\sin x)(\cos x)$$

$$= (2\sin x)(\cos x)$$

$$\frac{d}{dx}x^2 = 2x \qquad \neq 2\sin x$$

$$\sin x\cos x = \cos x\sin x$$

$$z = z$$

$$z + z = 2z$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x =$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$= \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{x} =$$

$$= \frac{x\cos x - \sin x}{x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(1+x^2)^2 = \frac{d}{dx}(1+x^2)(1+x^2) =$$

$$2x(1+x^2) + (1+x^2)2x$$

$$= 4x(1+x^2)$$

- Recall

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^2 x = 2\sin x\cos x.$$

$\sin 2x$
$\cos 2x$
$\cos x^2$

What about

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x^2 = 2x\cos x^2$$

- We'll discuss tomorrow…