Math 1210–23 Notes of 1/29/24

- So far we know these differentiation rules:
- **Power Rule:**

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}}$$

where n is a positive integer.

- **Sum Rule:**

$$\boxed{(f+g)' = f' + g'}$$

- **Constant Multiple Rule:**

$$\boxed{(kf)' = k(f')}$$

where k is a constant.

- With the above we can differentiate any polynomial!
- We also have these two miscellaneous rules:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x} = \frac{-1}{x^2}.$$

- But we need more rules!

More Differentiation Rules

The Product Rule

- The derivative of the product does not in general equal the product of the derivatives!
- Instead:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} f(x)g(x) = f'(x)g(x) + f(x)g'(x)},$$

or, more briefly

$$\boxed{(uv)' = u'v + uv'}$$

- One way to remember the product rule is to note that each of the two term can be obtained by considering one of the factors constant.

- Example: Compute

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2+2)(x^3+1)$$

in two different ways:

- Example: Compute

$$\frac{\mathrm{d}}{\mathrm{d}x}(x+\sqrt{x})(x^2+x)$$

- Why does the product rule work?
- We go back to the definition of the derivative and apply the Main Limit Theorem (the limit of the sum is the sum of the limits, etc.).

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f(x)g(x) &= \lim_{h\longrightarrow 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\longrightarrow 0} \frac{1}{h}\Big(f(x+h)g(x+h) - f(x)g(x+h) \\
&\qquad + f(x)g(x+h) - f(x)g(x)\Big) \\
&= \lim_{h\longrightarrow 0} \frac{f(x+h)g(x+h)-f(x)g(x+h)}{h} \\
&\qquad + \lim_{h\longrightarrow 0} \frac{f(x)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\longrightarrow 0} \frac{f(x+h)-f(x)}{h} \times \lim_{h\longrightarrow 0} g(x+h) \\
&\qquad + f(x) \lim_{h\longrightarrow 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}$$

- We'll see much more of the product rule when we learn how to differentiate functions other than polynomials.

The Quotient Rule

- The quotient rule is:

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$$

- Examples:

$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1+x^2} =$

$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x-1}{1+x^2} =$

$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x+3}{x+4} =$

$\frac{\mathrm{d}}{\mathrm{d}x}x^{-n} =$

- Why does the quotient rule work?
- Again, we do algebra, and apply the main limit theorem.

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \lim_{h\longrightarrow 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$\begin{aligned}
&= \lim_{h\longrightarrow 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \\
&= \lim_{h\longrightarrow 0}\frac{1}{g(x+h)g(x)} \\
&\quad \times \lim_{h\longrightarrow 0}\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h} \\
&= \lim_{h\longrightarrow 0}\frac{1}{g(x+h)g(x)} \\
&\quad \times \lim_{h\longrightarrow 0}\frac{f(x+h)g(x) - f(x)g(x)}{h} \\
&\quad \times \lim_{h\longrightarrow 0}\frac{f(x)g(x) - f(x)g(x+h)}{h} \\
&= \lim_{h\longrightarrow 0}\frac{1}{g(x+h)g(x)} \\
&\quad \times \lim_{h\longrightarrow 0}\frac{\big(f(x+h) - f(x)\big)g(x)}{h} \\
&\quad \times \lim_{h\longrightarrow 0}\frac{f(x)\big(g(x) - g(x+h)\big)}{h}
\end{aligned}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Derivatives of Trig Functions

- Can you guess the derivative of the sin function?

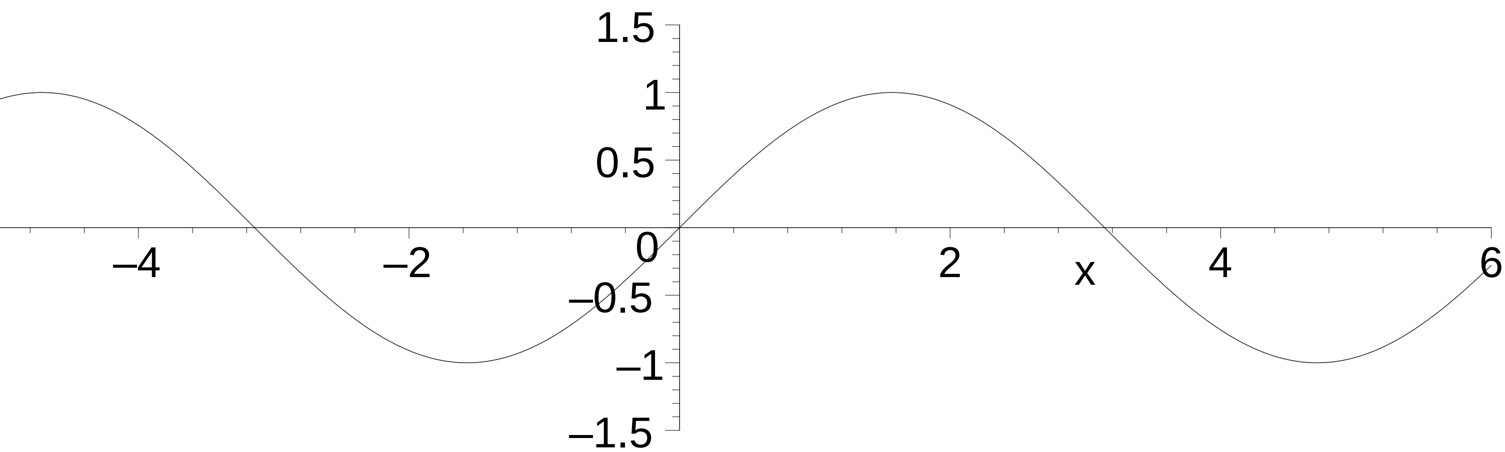

Figure 1. The Sine Function.

- Recall the graphs of sine and cosine:

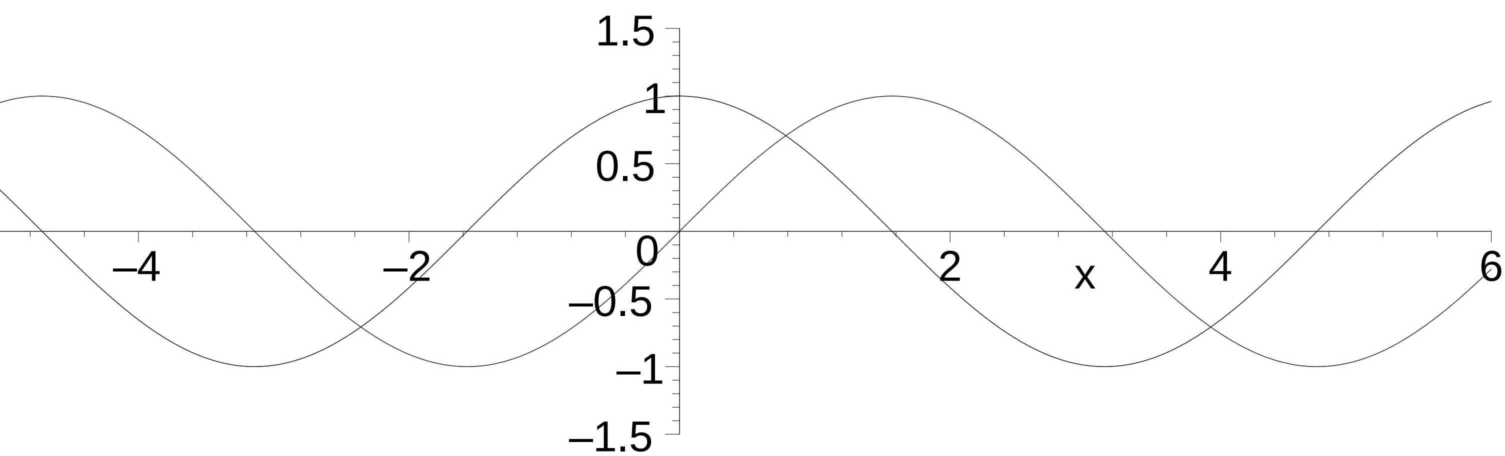

Figure 2. Graphs of sine and cosine.

- Yes indeed!

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x.$$

- Need trig identity:

$$\sin(x+h) = \sin x \cos h + \cos x \sin h.$$

- Also remember these limits:

$$\lim_{h\longrightarrow 0}\frac{\cos x - 1}{x} = 0 \quad \text{and} \quad \lim_{h\longrightarrow 0}\frac{\sin x}{x} = 1.$$

- Again, we go back to the definition and use the main limit theorem:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sin x &= \lim_{h\longrightarrow 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\longrightarrow 0}\frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\longrightarrow 0}\sin x \times \frac{\cos h - 1}{h} + \lim_{h\longrightarrow 0}\cos x \frac{\sin h}{h} \\
&= \sin x \underbrace{\lim_{h\longrightarrow 0}\frac{\cosh -1}{h}}_{=0} + \cos x \underbrace{\lim_{h\longrightarrow 0}\frac{\sin h}{h}}_{=1} \\
&= \cos x.
\end{aligned}$$

- Proceeding similarly (Textbook, page 115) we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x.$$

- More examples:

$$\frac{\mathrm{d}}{\mathrm{d}x}x^2\sin x =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^2 x =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{x} =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(1+x^2)^2 =$$

- Recall

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^2 x = 2\sin x\cos x.$$

What about

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x^2 =$$

- We'll discuss tomorrow…