Math 1210-23 Notes of 1/26/24

2.2 More on Derivatives

- Recall the definition of the derivative

$$f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- f' (f-prime) is the derivative of f (with respect to x).
- Geometrically:

- $f'(x)$ is the slope of the tangent at the point $\big(x, f(x)\big)$
- The slope of the secant approaches that of the tangent.

- The limit in the definition of the derivative can be written in many different ways, for example:

$$f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}$$

(the standard notation)

$$= \lim_{s \longrightarrow 0} \frac{f(x+s) - f(x)}{s}$$

(We don't have to use h as the variable)

$$= \lim_{\alpha \longrightarrow 0} \frac{f(x+\alpha) - f(x)}{\alpha}$$

(but we couldn't use x as the variable)

$$= \lim_{z \longrightarrow x} \frac{f(z) - f(x)}{z - x}$$

(same as the standard if $h = z - x$)

$$= \lim_{x_2 \longrightarrow x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

(where $x = x_1$)

- Limits, and derivatives, may not exist. Examples:
- $f(x) = |x|$ at $x = 0$
- $f(x) = \sqrt{|x|}$ at $x = 0$
- $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{else} \end{cases}$

Terminology

- Computing the derivative of f is called **differentiating** f.
- If $f'(x)$ exists then f is **differentiable** at x.
- If $f'(x)$ exists for all x in its domain then f is said to be differentiable.

Differentiability implies Continuity

- Remember that f is continuous at $x = c$ if

$$\lim_{x \longrightarrow c} f(x) = f(c)$$

- Important fact: if a function is differentiable at x then it is continuous at x.
- In other words

$$\text{f'(c) exists} \qquad \Longrightarrow \qquad \lim_{x \longrightarrow c} f(x) = f(c).$$

- Here is the argument from the textbook (Theorem A, page 102). It involves the main limit theorem (limit of the product is the product of the limits, etc.).
- Write

$$f(x) = f(c) + \frac{f(x) - f(c)}{x - c}(x - c) \qquad (x \neq c)$$

- Then

$$\lim_{x\longrightarrow c} f(x) = \lim_{x\longrightarrow c}\left(f(c) + \frac{f(x)-f(c)}{x-c}(x-c)\right)$$

$$= \lim_{x\longrightarrow c} f(c) + \underbrace{\lim_{x\longrightarrow c}\frac{f(x)-f(c)}{x-c}}_{=f'(c)} \times \underbrace{\lim_{x\longrightarrow c}(x-c)}_{=0}$$

(Note that all limits on the right exist)

$$= f(c) + f'(c)\times 0$$

$$= f(c)$$

More Notation

- There are very many notations for the derivative. You want to be comfortable with the notations given on the next page. We'll use each of them (except perhaps the last) at some stage or other.
- The Greek letter Δ (capital Delta) is often used to denote differences:

$$\begin{aligned} y &= f(x) \\ x &= x_1 \\ \Delta x &= x_2 - x_1 \\ \Delta f &= f(x_2) - f(x_1) \end{aligned}$$

$$f'(x) = \lim_{\Delta x \longrightarrow 0} \frac{\Delta f}{\Delta x}$$

$$= f'(x)$$

(pronounced “f-prime of x”)

$$= \frac{\mathrm{d}f}{\mathrm{d}x} \quad (\text{or} \quad \mathrm{d}f/\mathrm{d}x)$$

(pronounced “dee-f-dee-x”)

$$= \mathrm{D}f$$

(pronounced “dee-f”)

$$= \mathrm{D}y$$

(pronounced “dee-y”)

$$= \mathrm{D}_x y$$

(pronounced “dee-x of y”)

$$= \frac{\mathrm{d}}{\mathrm{d}x} y$$

(pronounced “dee-dee-x of y”)

$$= \frac{\mathrm{d}y}{\mathrm{d}x}$$

(pronounced “dee-y dee-x”)

Differentiation Rules

- Differentiation rules are formulas for computing derivatives.
- We already computed the derivatives in the following Table:

$f(x)$	$f'(x)$
k	0
$mx+b$	m
x^2	$2x$
x^3	$3x^2$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\frac{1}{x}$	$-\frac{1}{x^2}$

We need to get more systematic.

The Power Rule

- Suppose n is a positive integer. Then

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}}$$

Linearity

- Suppose k is a constant.
- We have the **constant multiple rule** (Theorem D)

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[kf(x)\right] = k\frac{\mathrm{d}}{\mathrm{d}x}f(x)}$$

and the **sum rule** (Theorem E)

$$\boxed{\frac{d}{\mathrm{d}x}\big(f(x)+g(x)\big) = \frac{\mathrm{d}}{\mathrm{d}x}f(x) + \frac{\mathrm{d}}{\mathrm{d}x}g(x)}$$

- These are easy to see, see the textbook. The rules can be written more succinctly as

$$(kf)' = kf' \quad \text{and} \quad (f+g)' = f' + g'$$

or

$$D(kf) = kDf \quad \text{and} \quad D(f+g) = Df + Dg.$$

- D is an operator, the **differentiation operator**. An operators associates a function with a function (in this case, a function with its derivative).

- Any operator F that satisfies, for constants c_1 and c_2 and functions f_1 and f_2,

$$F(c_1 f_1 + c_2 f_2) = c_1 F f_1 + c_2 F f_2$$

  is said to be linear.
- **Differentiation** $(F = D)$ **is Linear**.
- With the above rules we can differentiate any polynomial!

Polynomial Examples

- $D_x(4x^3 - 12x^2 + 7x + 4) =$

- $D_x(10x^9 + 5x^7) =$

- $D_x(x + 2)(x + 3) =$

- $D_x(x + 2)^2 =$

- $D_x(x + 2)^3 =$

- How about $D_x(1 + x^2)^2$?

The Product Rule

- The limit of the sum is the sum of the limits, the limit of the product is the product of the limits, the derivative of the sum is the sum of the derivatives, so is the derivative of the product the product of the derivatives?