1.4 Limits Involving Trig Functions

- Recall the definition of angles, sine, and cosine

![Figure 1. Sine and Cosine.](image)

- Recall

\[ \tan t = \frac{\sin t}{\cos t} \]
Figure 2. Graphs of sin, cos, tan, x.

- It’s clear from the graph that

\[
\lim_{t \to c} \sin t = \sin c \quad \text{and} \quad \lim_{t \to c} \cos t = \cos c
\]

- There is a more rigorous discussion (Theorem A, page 74) in the textbook.
• We will need the following limits:

\[
\lim_{t \to 0} \frac{\sin t}{t} = 1 \quad \text{and} \quad \lim_{t \to 0} \frac{1 - \cos t}{t} = 0.
\]

• This is the contents of Theorem B on page 75.

• We already saw the first statement:

![Figure 3](image)

**Figure 3.** \(\lim_{t \to 0} \frac{\sin t}{t} = 1\).

• Let’s look a little closer. Recall

**The Squeeze Theorem.** Suppose \(f\), \(g\), and \(h\) are functions such that

\[
f(x) \leq g(x) \leq h(x)
\]

for all \(x\) near \(c\) except possibly at \(x = c\). Also assume that

\[
\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L
\]

Then

\[
\lim g(x) = L
\]
Now recall the area of a sector of a circle:
\[
\text{area} = \frac{t}{2} r^2.
\]

Apply to the sector shown here:

\[\begin{align*}
O &= (0, 0), & A &= (1, 0), & B &= (\cos t, 0), & P &= (\cos t, \sin t), & C &= (\cos^2 t, \cos t \sin t)\\
\text{and} & \\
 f(t) &= \text{area}(\text{sector}OBC) = \frac{t}{2} \cos^2 t \\
g(t) &= \text{area}(\triangle OBP) = \frac{1}{2} \cos t \sin t \\
h(t) &= \text{area}(\text{sector}OAP) = \frac{t}{2}.
\end{align*}\]

Clearly
\[
f(t) \leq g(t) \leq h(t)
\]
or
\[
\frac{t}{2} \cos^2 t \leq \frac{1}{2} \cos t \sin t \leq \frac{t}{2}.
\]
Multiply with 2 and divide by \( t \cos t \) to get

\[
\cos t \leq \frac{\sin t}{t} \leq \frac{1}{\cos t}
\]

Clearly

\[
\lim_{t \to 0} \cos t = \lim_{t \to 0} \frac{1}{\cos t} = 1
\]

and, by the Squeeze Theorem

\[
\lim_{t \to 0} \frac{\sin t}{t} = 1.
\]

- This implies that

\[
\lim_{t \to 0} \frac{1 - \cos t}{t} = 0:
\]

\[
\lim_{t \to 0} \frac{1 - \cos t}{t} = \lim_{t \to 0} \frac{(1 - \cos t)(1 + \cos t)}{t + (1 + \cos t)} = \lim_{t \to 0} \frac{1 - \cos^2 t}{t(1 + \cos t)} = \lim_{t \to 0} \frac{\sin^2 t}{t(1 + \cos t)} = \lim_{t \to 0} \frac{\sin t}{t} \cdot \lim_{t \to 0} \frac{\sin t}{1 + \cos t} = 1 \cdot 0 = 0
\]
More Limits

\[
\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = \lim_{\theta \to 0} \frac{\sin \theta}{\cos \theta \cdot \theta} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta \cos \theta}
\]

\[
= \lim_{\theta \to 0} \frac{\sin \theta}{\cos \theta} \cdot \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \cdot 1 = 1
\]

\[
\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\sin \left(\frac{x \cdot \pi}{180}\right)}{\frac{x \cdot \pi}{180}} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \frac{\pi}{180} \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \frac{\pi}{180} \cdot 1 = \frac{\pi}{180}
\]

\[
\lim_{\theta \to 0} \frac{\tan 5\theta}{\sin 2\theta} = \lim_{\theta \to 0} \frac{(\tan 5\theta) 2\theta}{5\theta \cdot \sin 2\theta \cdot 2\theta}
\]

\[
= \lim_{\theta \to 0} \frac{5\theta}{2\theta} \cdot \lim_{\theta \to 0} \frac{2\theta}{\sin 2\theta} \cdot \lim_{\theta \to 0} \frac{\tan 5\theta}{5\theta}
\]

\[
= \frac{5}{2} \cdot 1 \cdot 1 = \frac{5}{2}
\]
1.6 Continuity

- Remember
  Concept → Definition → Properties → Work

- Intuition: A function is **continuous** if its graph can be drawn without lifting the pencil.

- Continuous or not:

  \[ f(x) = x^2 \]
  \[ f(x) = \sin x, \cos x \]
  \[ f(x) = |x| \]
  \[ f(x) = \frac{1}{x} \]
  \[ f(x) = \tan x \]
  \[ f(x) = \begin{cases} 0 & \text{if } x = \pi \\ 1 & \text{else} \end{cases} \]
  \[ f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases} \]