2.1-2 The Derivative

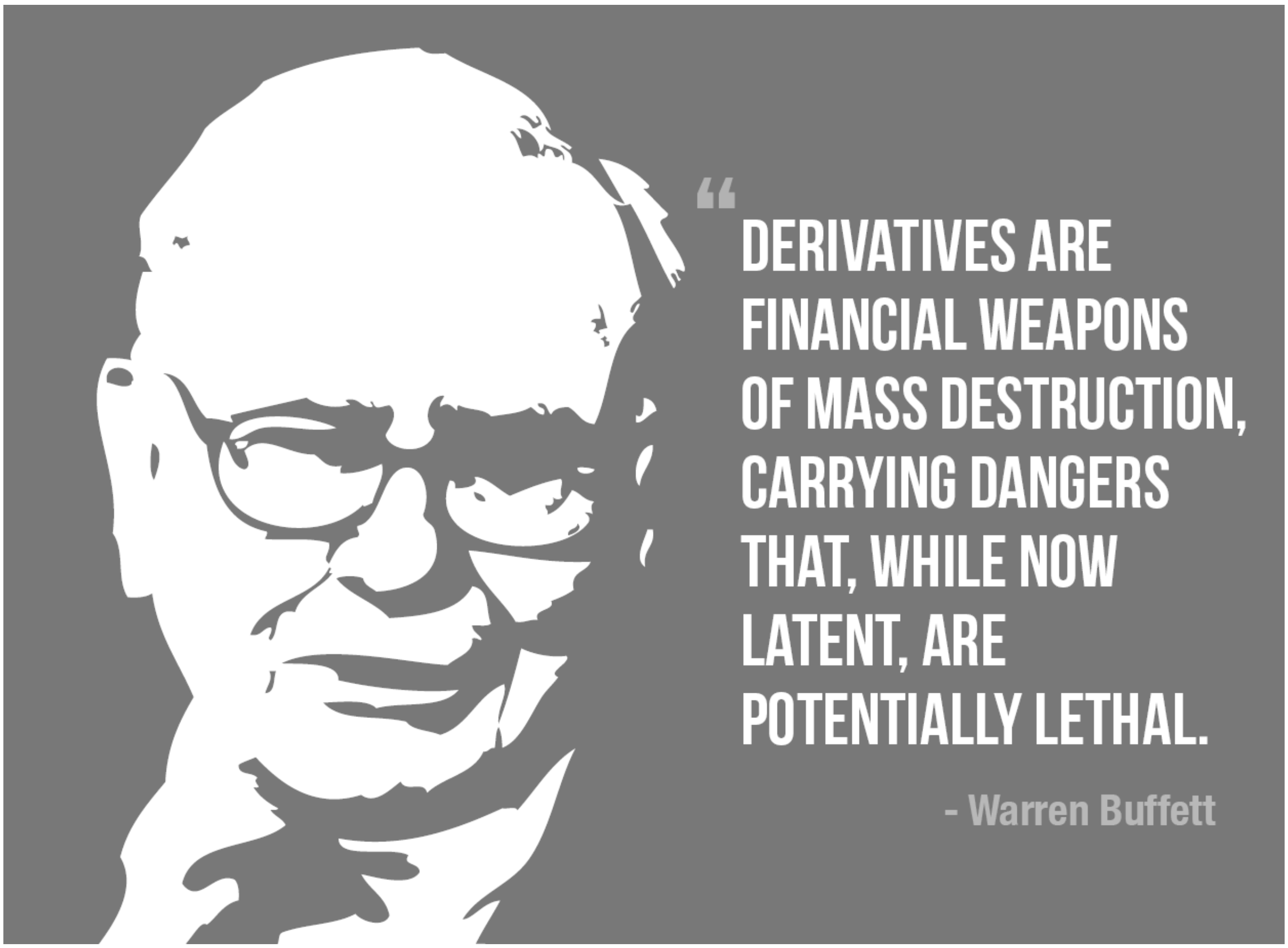


Figure 1. Warren Buffett on Derivatives.

- We need a fresh start.
- Remember

  Concept $\longrightarrow$ Definition $\longrightarrow$ Properties $\longrightarrow$ Work
- Recall our discussion of velocity:
- $s(t)$ = location at time t
- $v(t)$ = velocity at time t

$$v(t) \approx \frac{s(t+h)-s(t)}{h}$$

- We consider what happens as h approaches zero.
- Well, now that we know about limits, we can define:

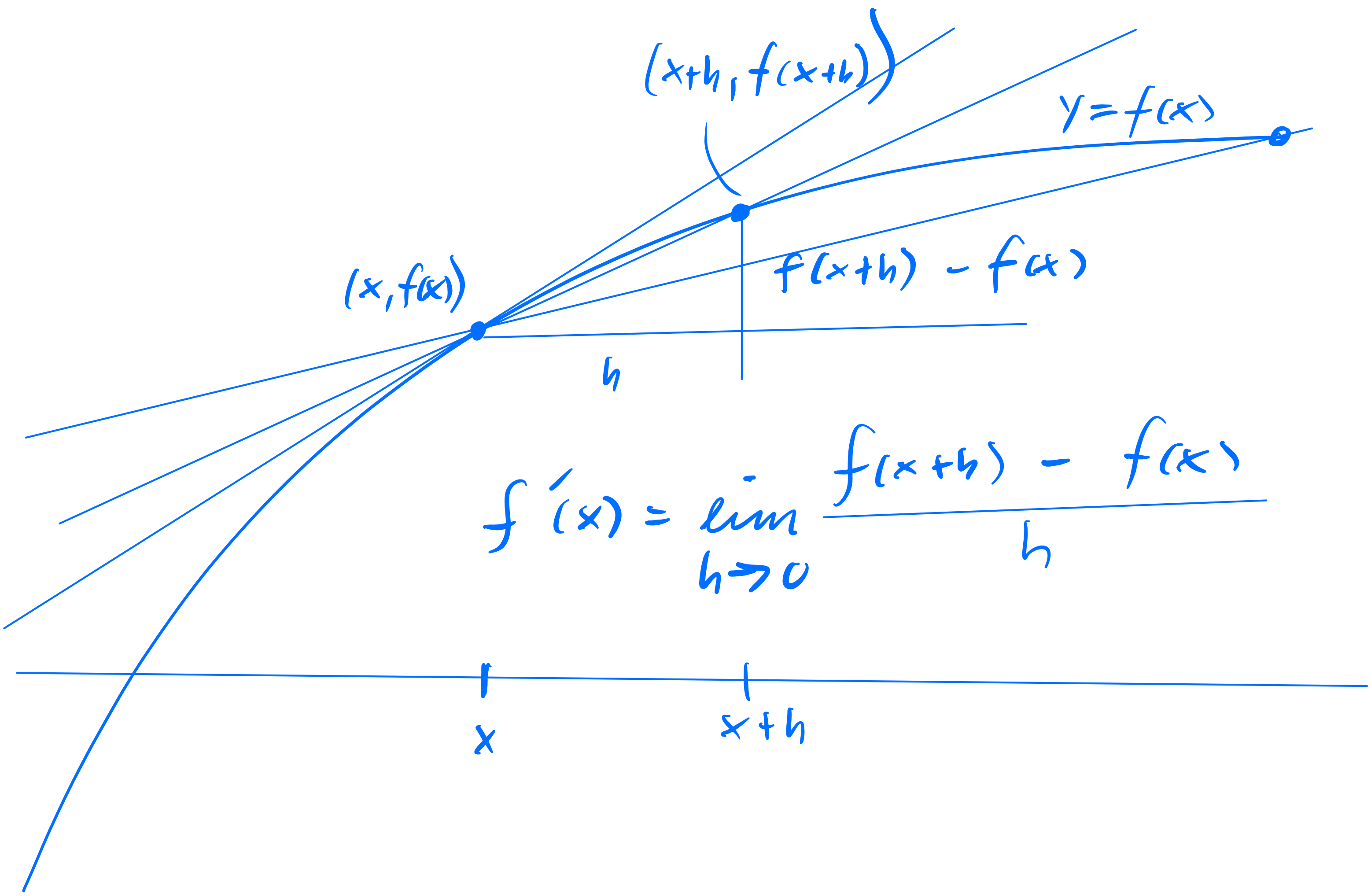
$$v(t) = \lim_{h \longrightarrow 0} \frac{s(t+h)-s(t)}{h}$$

- we say that velocity is the **derivative** of location.
- In general, given a function f, its derivative f' is another function defined by

$$f'(t) = \lim_{h \longrightarrow 0} \frac{f(t+h)-f(t)}{h} \qquad \frac{0}{0}$$

- This is the heart of Calculus. It's crucial to understand this definition and concept both algebraically and geometrically.

The Geometry



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

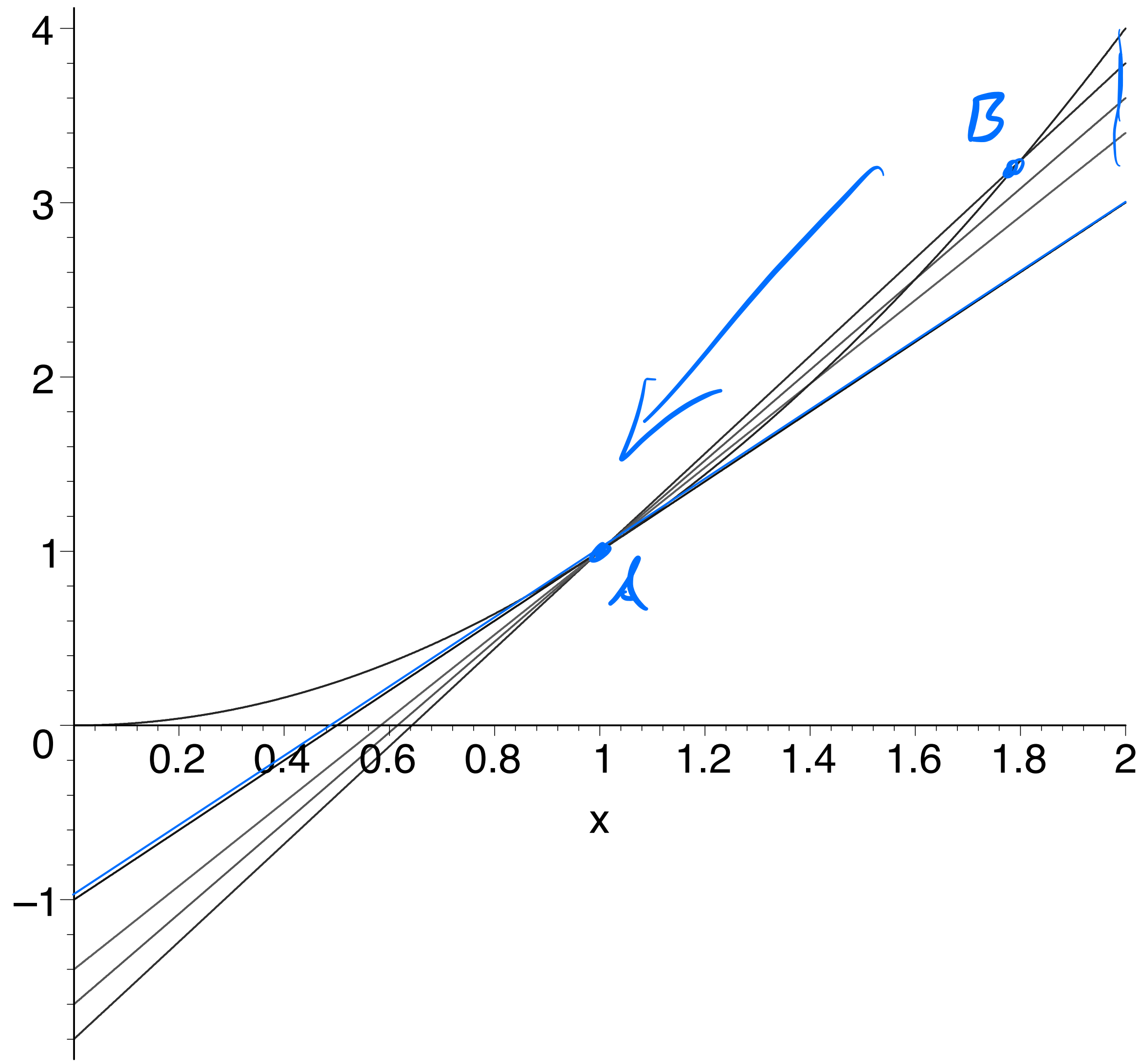


Figure 2. Secants and Tangent.

- The derivative gives the slope of the tangent (line) as the limit of the slopes of the secants (lines).
- The derivative is a function!

- For the rest of today we'll do some examples.
- $f(x) = x^2$ (Review)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2hx + h^2}{h} = \lim_{h\to 0} (2x + h)$$

$$= 2x = f'(x)$$

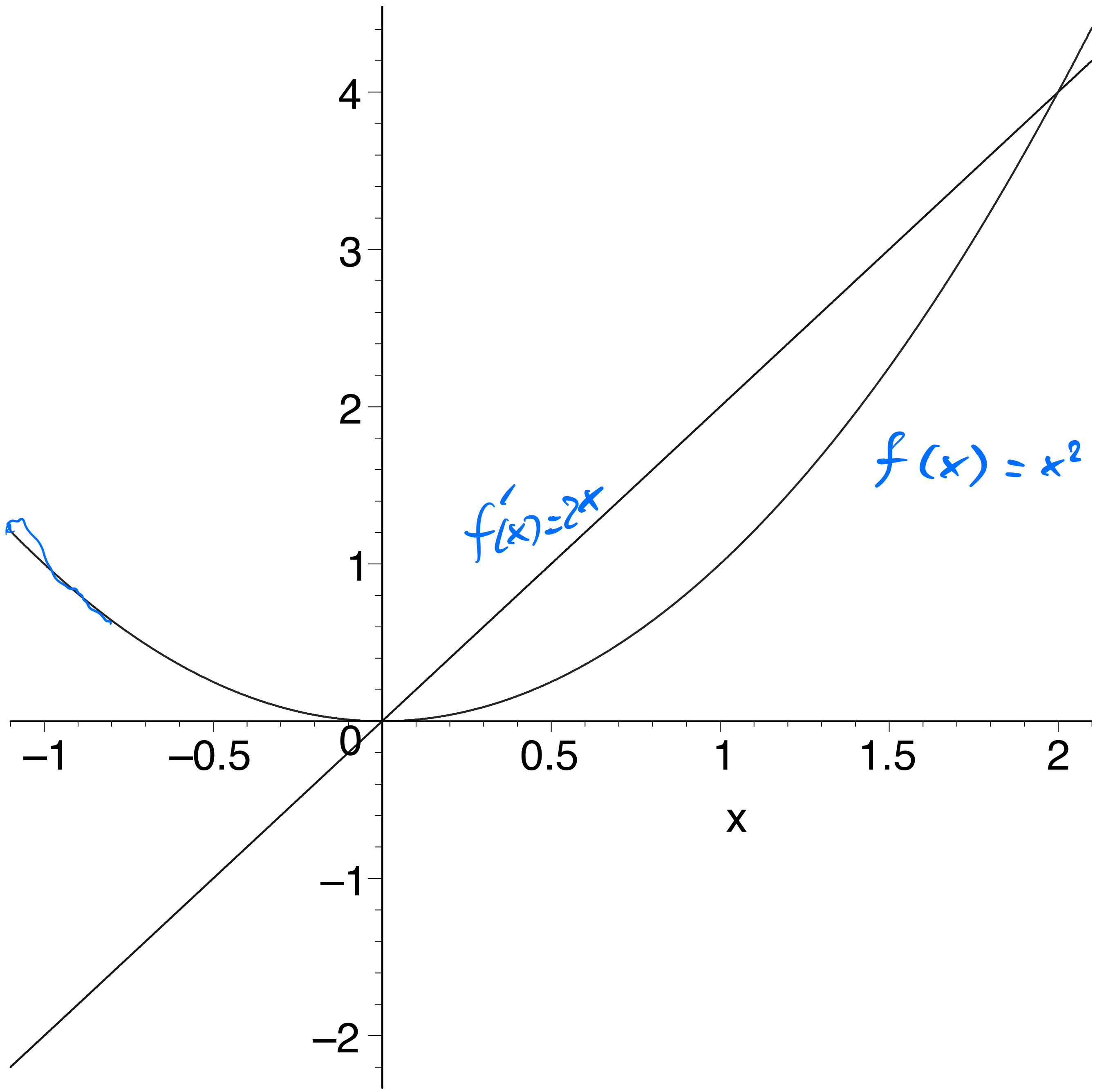


Figure 3. $f(x) = x^2$ and $f'(x) = 2x$.

- General principle: apply a new concept to a familiar context.
- The derivative of a constant should be zero:

$$f(x) = c$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h}$$

$$= \lim_{h \to 0} 0 = 0 \checkmark$$

- The derivative of a linear function should be constant.

$$f(x) = mx + b \qquad f'(x) =$$

$$f'(x) = \lim_{h \to 0} \frac{m(x+h) + b - (mx+b)}{h}$$

$$= \lim_{h \to 0} \frac{mx + mh + b - mx - b}{h}$$

$$= \lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} m = m$$

- $f(x) = x^3$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$(x+h)^3 = (x+h)^2 (x+h)$$

$$= (x^2 + 2hx + h^2)(x+h)$$

$$= x^3 + 2hx^2 + h^2 x$$

$$+ hx^2 + 2h^2 x + h^3$$

$$= x^3 + 3hx^2 + 3h^2 x + h^3$$

$$f'(x) = \lim_{h \to 0} \frac{3hx^2 + 3h^2 x + h^3}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3hx + h^2 = 3x^2$$

HOT = higher order terms

$f(x) = x^4$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h}$$

$$= \lim_{h \to 0} \frac{x^4 + 4hx^3 + HOT - x^4}{h}$$

$(x+h)^4 = (x+h)(x+h)(x+h)(x+h)$

$= 4x^3$

$$f(x) = x^n \rightarrow \frac{x^n + nhx^{n-1} + HOT - x^n}{h}$$

$$\rightarrow nx^{n-1}$$

$$f'(x) = nx^{n-1}$$

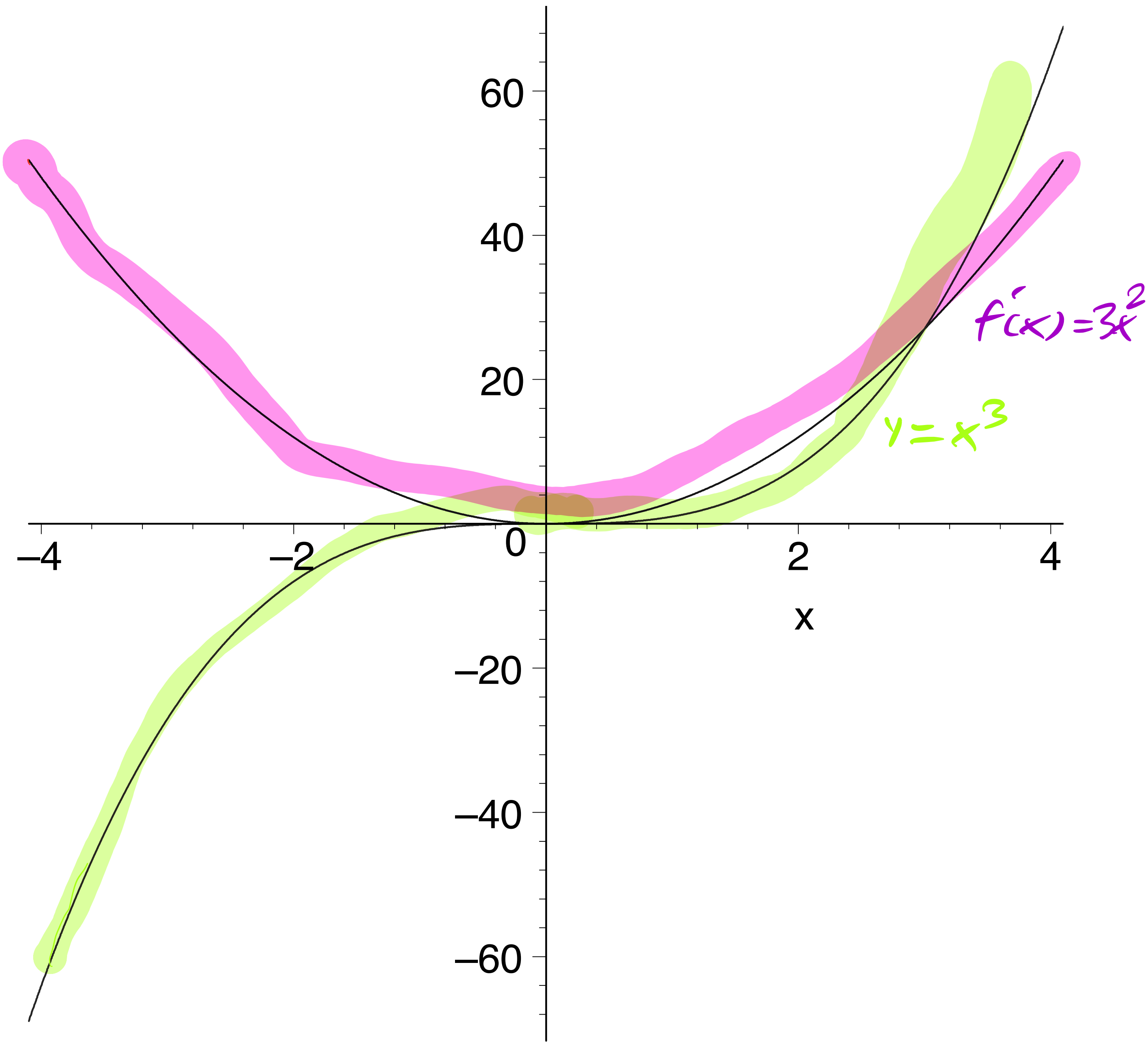


Figure 4. $f(x) = x^3$ and $f'(x) = 3x^2$.

- Note the discrepancy in the horizontal and vertical scales.

$$(a+b)(a-b) = a^2 - b^2$$

- $f(x) = \sqrt{x}$ $= x^{1/2}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-1/2}$$

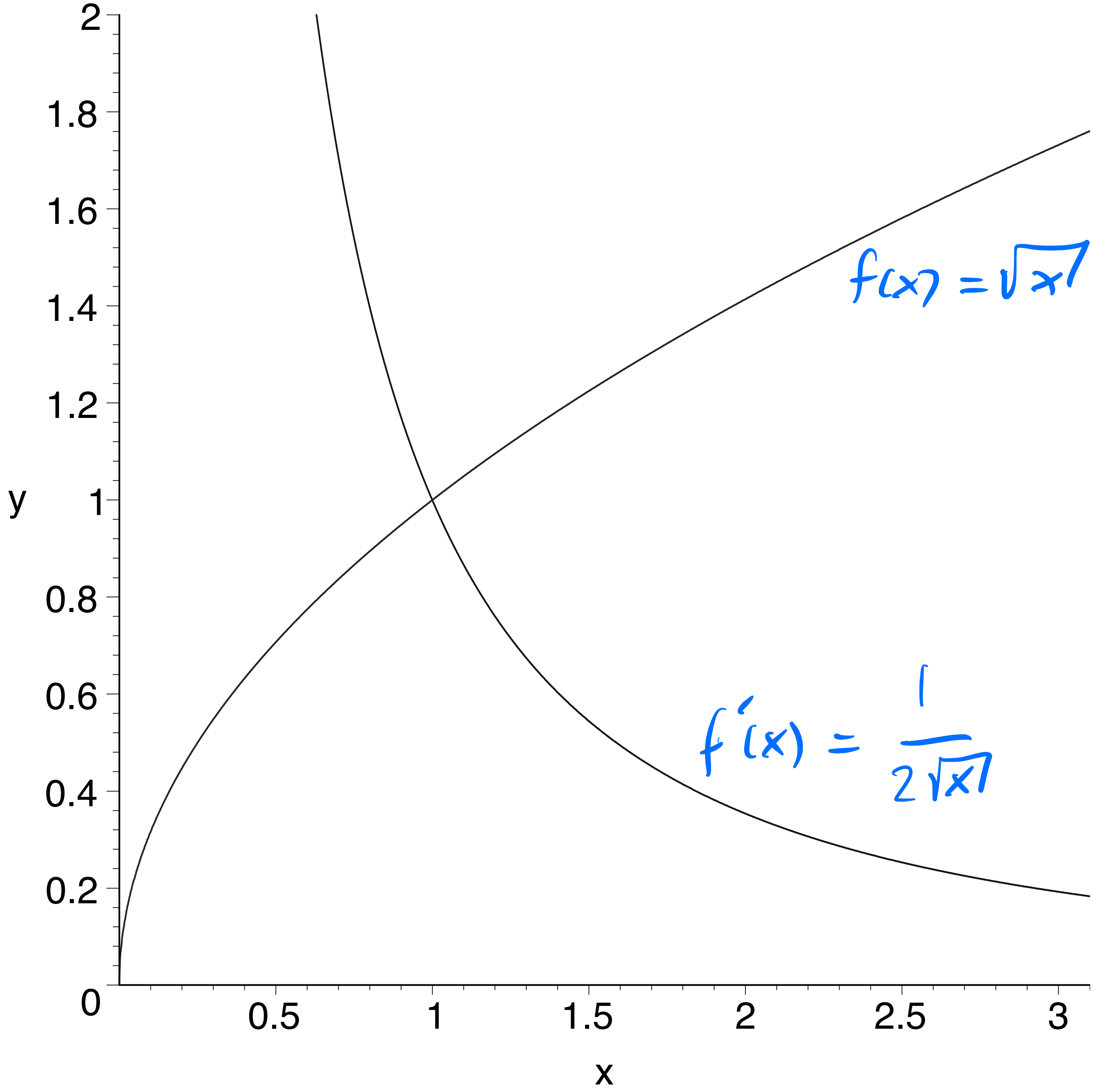


Figure 5. $f(x) = \sqrt{x}$ and $f'(x) = \frac{1}{2\sqrt{x}}$.

- $f(x) = \frac{1}{x} = x^{-1}$ $\qquad f'(x) = -x^{-2}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x-(x+h)}{(x+h)x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x - x - h}{(x+h)x}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{-h}{(x+h)x}\right) = \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$= -\frac{1}{x^2}$$

$$f(x) = x^p \qquad f'(x) = p\,x^{p-1}$$

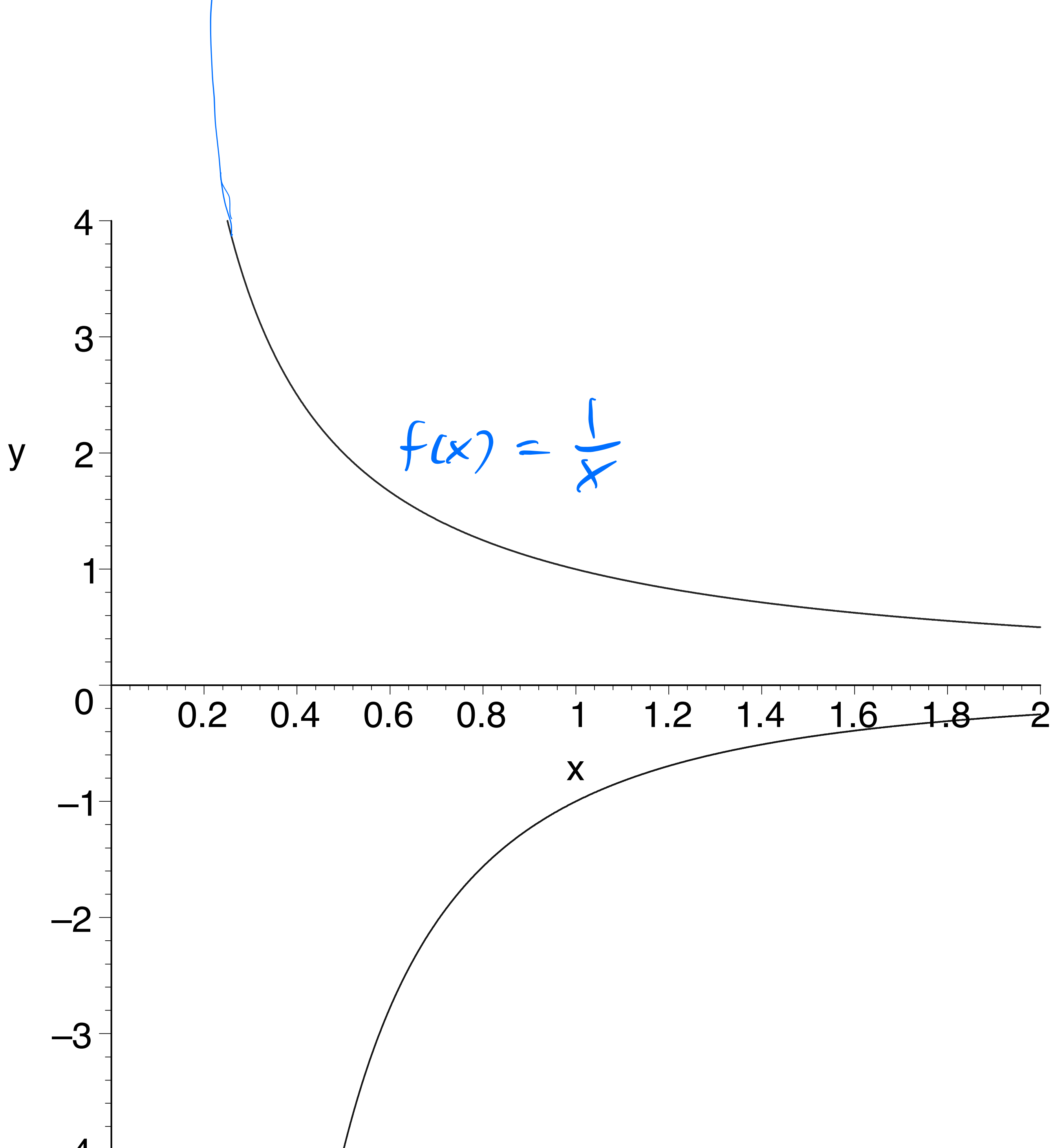


Figure 6. $f(x) = \frac{1}{x}$ and $f'(x) = -\frac{1}{x^2}$.

- $f(x) = 3x^2$

$$f(x) = \lim_{h\to 0} \frac{3(x+h)^2 - 3x^2}{h}$$

$$= 3 \cdot \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= 3 \cdot 2x = 6x$$

$$f(x) = k g(x) \qquad f'(x) = k g'(x)$$

- $f(x) = x^2 + x^3$

$$f'(x) = 2x + 3x^2$$

$$f(x) = x^p \qquad f'(x) = p x^{p-1}$$

$$(f(x) + g(x))' = f'(x) + g'(x)$$

$$(kf(x))' = k f'(x)$$