1.3, 1.5, Working with Limits

• Recall Procedure:
  Concept $\rightarrow$ Definition $\rightarrow$ Properties $\rightarrow$ Work

• Our Definition: We say that the limit of $f(x)$ as $x$ approaches $c$ equals $L$, or

$$\lim_{x \to c} f(x) = L$$

if for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

• $\epsilon$ is the lower case Greek letter epsilon, and $\delta$ is the lower case Greek letter delta.
Properties:

Main Limit Theorem

- See textbook, page 68.
- Let \( n \) be a positive integer, \( k \) a constant, and \( f \) and \( g \) functions that have limits at \( c \). Then:

1. \( \lim_{x \to c} k = k \).

2. \( \lim_{x \to c} x = c \).

3. \( \lim_{x \to c} kf(x) = k \lim_{x \to c} f(x) \).

4. \( \lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) \).

5. \( \lim_{x \to c} (f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) \).

6. \( \lim_{x \to c} (f(x) \cdot g(x)) = \left( \lim_{x \to c} f(x) \right) \cdot \left( \lim_{x \to c} g(x) \right) \).

7. \( \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \) provided \( \lim_{x \to c} g(x) \neq 0 \).

8. \( \lim_{x \to c} (f(x))^n = \left( \lim_{x \to c} f(x) \right)^n \).

9. \( \lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} \) provided \( f(x) \geq 0 \) when \( n \) is even.
• Consequence of Main Limit Theorem:

\[ \lim_{x \to c} f(x) = f(c) \]

if \( f \) is a polynomial or a rational function with a non-zero denominator at \( x = c \).

• Example

\[ \lim_{x \to 1} \frac{x + 4}{x^2 + 1} = \frac{5}{2} \]
• subtle point, and frequent source of errors: when combining two functions the limit may exist even if the individual limits do not.

• simple example:

\[ f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = -\frac{1}{x} \]

\[
\lim_{x \to 0} \frac{1}{x} \quad \text{DNE (does not exist)}
\]

\[
\lim_{x \to 0} (f(x) + g(x)) = \lim_{x \to 0} \left( \frac{1}{x} - \frac{1}{x} \right)
\]

\[
= \lim_{x \to 0} 0 = 0
\]

\[
\neq \lim_{x \to 0} f(x) + \lim_{x \to 0} g(x)
\]
• We proved item 4 of the main limit theorem in class.

• Proof of the other parts is in the textbook in section 1.3.

• Another major fact is

**The Squeeze Theorem.** Suppose \( f, g, \) and \( h \) are functions such that

\[
f(x) \leq g(x) \leq h(x)
\]

for all \( x \) near \( c \) except possibly at \( x = c \). Also assume that

\[
\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L
\]

Then

\[
\lim_{x \to c} g(x) = L
\]

• This is geometrically very plausible. See Figure 2 on page 72.

• Exercise for the ambitious: Prove the Squeeze Theorem using the \( \epsilon - \delta \) definition of limits.
• Example:

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0.$$ 

**Figure 1.** Graph of $y = \sin \frac{1}{x}$.

**Figure 2.** Graph of $y = x \sin \frac{1}{x}$ and $y = \pm |x|$. 

$h(x) = |x|$

$g(x) = x \sin \frac{1}{x}$

$f(x) = -|x|$
One Sided Limits

- the limit properties we have discussed so far also apply to one sided limits.

- Examples:

- Recall
  \[ [x] = \text{the greatest integer } \leq x. \]

- For example:

\[
\lim_{x \to 2^-} [x]^2 + 1 = 2
\]
\[
\lim_{x \to 2^+} [x]^2 + 1 = 5
\]

\[
\lim_{x \to 2^-} [x] = 1
\]
\[
\lim_{x \to 2^+} [x] = 2
\]
Limits and Infinity

• There are also the concepts of "limits at infinity" and "infinite limits".

• Some examples:

\[
\lim_{x \to \infty} \frac{1}{1 + x^2} = 0
\]

\[
\lim_{x \to \infty} \arctan(x) = \frac{\pi}{2}
\]

\[
\lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2}
\]

• The next two examples are written as "infinite" limits, but actually are examples of non-existent limits.

\[
\lim_{x \to 1^+} \frac{x}{x - 1} = +\infty \quad \text{limit does not exist} \quad \frac{x}{x-1} \text{ grows without bounds}
\]

\[
\lim_{x \to 1^-} \frac{x}{x - 1} = -\infty \quad \text{singularity at } x = 1 \quad \frac{x}{x-1} \text{ blows up}
\]

• Also note that

\[
\lim_{x \to \infty} \frac{x}{x - 1} = 1
\]

\[
\lim_{x \to 0} \frac{1}{x^2} = +\infty
\]
More Examples

\[ \lim_{x \to -\infty} \frac{1}{1 + x^2} = 0 \]

\[ \lim_{x \to \infty} \frac{x^3 + 3x^2 - \pi x + 4}{3x^3 + 2x^2 - 4x + 1} = \frac{1}{3} \]

\[ \lim_{x \to \infty} \frac{x + \ln x}{x} = ( \text{undefined} ) \]

- View ahead to Math 1220 (Ex. 4, page 74), limits of sequences: Suppose

\[ a_n = \sqrt{\frac{n + 1}{n + 2}}, \quad n = 1, 2, 3, \ldots \]

Then

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \sqrt{\frac{n+1}{n+2}} = \]

\[ a_1 = \sqrt{\frac{2}{3}} \quad a_2 = \sqrt{\frac{2}{4}} \quad a_3 = \sqrt{\frac{3}{5}} \]
Yet more Examples

\[ \lim_{{x \to 2^+}} \frac{x - 1}{x - 2} = +\infty \]

\[ \lim_{{x \to 2^-}} \frac{x - 1}{x - 2} = -\infty \]

\[ \lim_{{x \to 1}} \frac{x - 1}{x - 2} = 0 \]

\[ \lim_{{x \to \infty}} \frac{x^2}{2x} = 0 \]

\[ \lim_{{x \to -\infty}} \frac{x^2}{2x} = \infty \]

\[ \lim_{{n \to \infty}} \sum_{{k=0}}^{n} \frac{1}{2^k} = \]

\[ \frac{n}{2} \sum_{{k=0}}^{n} \frac{1}{2^k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \]

\[ \frac{1}{2} \sum_{{k=0}}^{n} \frac{1}{2^k} = \frac{3}{2} \cdot \frac{1}{2} + \frac{3}{2} \cdot \frac{1}{4} + \frac{3}{2} \cdot \frac{1}{8} + \ldots \]

\[ = \frac{3}{2} \cdot \frac{1}{2^0} + \frac{3}{2} \cdot \frac{1}{2^1} + \frac{3}{2} \cdot \frac{1}{2^2} + \ldots \]

\[ = 1 + \frac{3}{2} \cdot \frac{1}{2} + \frac{3}{2} \cdot \frac{1}{4} + \frac{3}{2} \cdot \frac{1}{8} + \ldots \]

\[ \approx 2 \]
Yet More Examples

\[ \lim_{x \to 1} \frac{x^2 + 6x - 7}{x - 1} = \]

\[ \lim_{x \to 7} \frac{x^2 + 6x - 7}{x - 1} = \]

\[ \lim_{x \to 0} \frac{x^2 + 6x - 7}{x - 1} = \]

\[ \lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2} = \]

\[ \lim_{x \to 3} \frac{x - 3}{x + 3} = \]

\[ \lim_{x \to 0} \frac{\sin(2x)}{x} = \]