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Lesson # 9 (10) Derived functors (cont)

(11) Adjoint functors and left/right exactness

Theorem: Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor between two abelian categories. If $\mathcal{A}$ has enough projectives, then

(1) Each L_nF is an additive functor.

(2) $L_*F = \{L_nF\}_{n\geq 0}$ form a homological δ-functor.

(3) L_*F is a universal δ-functor.

Proof: (1) We show first that L_nF is a functor.

$L_nF(id^A) := H_nF(id^{P_\cdot}) = H_n(id^{F(P_\cdot)}) = id^{H_nF(P_\cdot)} = id^{L_nF(A)}$.

Let $f: A' \longrightarrow A$ and $g: A \longrightarrow A''$ be morphisms in $\mathcal{A}$, and let $P_\cdot' \longrightarrow A'$, $P_\cdot \longrightarrow A$ and $P_\cdot'' \longrightarrow A''$ be proj resolutions.

If $f_\cdot : P_\cdot' \longrightarrow P_\cdot$ and $g_\cdot : P_\cdot \longrightarrow P_\cdot''$ are extensions of f and g then $g_\cdot f_\cdot : P_\cdot' \longrightarrow P_\cdot''$ is an extension of $gf: A' \longrightarrow A''$.

Therefore, $L_nF(gf) := H_nF(g_\cdot f_\cdot) = H_nF(g_\cdot)\, H_nF(f_\cdot)$ ... since H_n and F are functors. ... $= L_nF(g)\, L_nF(f)$.

L_nF is additive: Let $f_1, f_2 : A' \longrightarrow A$ be two morphisms and let $(f_1)_\cdot : P_\cdot' \longrightarrow P_\cdot$ and $(f_2)_\cdot : P_\cdot' \longrightarrow P_\cdot$ be two liftings; then $(f_1)_\cdot + (f_2)_\cdot$ is a lifting of $f_1 + f_2$. Therefore,

$$L_nF(f_1 + f_2) := H_nF((f_1)_\cdot + (f_2)_\cdot) = H_nF((f_1)_\cdot) + H_nF((f_2)_\cdot)$$
$$= L_nF(f_1) + L_nF(f_2).$$

(2) Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence in $\mathcal{A}$. By Horse shoe lemma there exists projective resolutions such that

$$\begin{array}{ccccccccc} 0 & \to & A' & \to & A & \to & A'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & P_\cdot' & \to & P_\cdot & \to & P_\cdot'' & \to & 0 \end{array}$$

is a commutative diagram with exact rows.

Since each $0 \to P_n' \to P_n \to P_n'' \to 0$ splits, we have an exact sequence $0 \to F(P_n') \to F(P_n) \to F(P_n'') \to 0$ for each $n \in \mathbb{Z}$. (it split in $\mathcal{B}$)

Therefore, $0 \to F(P_\cdot') \to F(P_\cdot) \to F(P_\cdot'') \to 0$ is an exact sequence of complexes.

We consider the long exact sequence of homologies and obtain

$$\cdots \to L_nF(A') \to L_nF(A) \to L_nF(A'') \xrightarrow{\delta_n} L_{n-1}F(A') \to \cdots$$

<u>The map δ_n is natural $\forall n$.</u> Let
$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r, "\alpha'"] \arrow[d, "f'"] & A \arrow[r, "\alpha''"] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & B' \arrow[r, "\beta'"] & B \arrow[r, "\beta''"] & B'' \arrow[r] & 0 \end{tikzcd}$$
be a morphism of short exact sequences.

We consider liftings of the morphisms f' and f'' to proj resolutions

$$\begin{tikzcd} 0 \arrow[r] & P_0' \arrow[r] \arrow[d, "\forall f_0'"'] & P_0 \arrow[r] \arrow[d, dashed, "\exists f_0"] & P_0'' \arrow[r] \arrow[d, "\forall f_0''"] & 0 \\ 0 \arrow[r] & Q_0' \arrow[r] & Q_0 \arrow[r] & Q_0'' \arrow[r] & 0 \end{tikzcd}$$

ETS to construct f_0 a lifting of f such that the diagram of projective resolutions is commutative. Indeed, it remains commutative after applying F and use that H_* is a homological δ-functor.

The construction of f_0 is done by induction.

The differential of P_0 is given by $d^{P_0} = \begin{pmatrix} d^{P_0'} & \theta^P \\ 0 & d^{P_0''} \end{pmatrix}$.

—— " —— Q_0 is given by $d^{Q_0} = \begin{pmatrix} d^{Q_0'} & \theta^Q \\ 0 & d^{Q_0''} \end{pmatrix}$

<u>Check</u> that $f_0 = \begin{pmatrix} f_0' & \Delta \\ 0 & f_0'' \end{pmatrix}$ where $\Delta : P'' \longrightarrow Q'[-1]$

satisfy: $\Delta_{i-1} d_i^{P''} - d_i^Q \Delta_i = \theta_i^Q f_i'' - f_{i-1}' \theta_i^P$

and $\beta' \eta' \Delta_0 = f \delta_0 - \gamma_0 f_0''$

$$\begin{tikzcd} A \arrow[r] & A'' \\ P_0 \arrow[u] \arrow[r] & P_0'' \arrow[u] \arrow[ul, "\gamma_0"'] \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r] & B'' \\ Q_0 \arrow[u] \arrow[r] & Q_0'' \arrow[u] \arrow[ul, "\delta_0"'] \end{tikzcd}$$

Δ is constructed by induction.

(3) <u>Remark</u>: If P is projective, then $L_nF(P) = 0 \ \forall n > 0$.

Let T_* be another homological δ-functor and let $\varphi_0 : T_0 \to F$

be given. We construct natural transformations $\varphi_n : T_n \longrightarrow L_nF$ that commute with δ_n, by induction.

<u>Construction of φ_n</u>, when φ_i is already defined (commuting with δ_i) for all $0 \leq i < n$:

Let A be in $\mathcal{A}$ and choose $P \twoheadrightarrow A$, P projective and consider the short exact sequence $0 \to K \to P \to A \to 0$. We obtain:

$$\begin{tikzcd}
\cdots \arrow[r] & T_n(K) \arrow[r] & T_n(P) \arrow[r] & T_n(A) \arrow[r, "\delta_n"] \arrow[d, dashed, "\exists!\, \varphi_n(A)"'] \arrow[dr, phantom, "\circledast"] & T_{n-1}(K) \arrow[r, "\beta"] \arrow[d, "\varphi_{n-1}(K)"] \arrow[dr, "\gamma"] & \cdots \\
\cdots\cdots & & L_nF(P) \arrow[r] \arrow[d, equal] & L_nF(A) \arrow[r, hook, "i"'] & L_{n-1}F(K) \arrow[r, "\alpha"'] & \cdots \\
& & 0 & & &
\end{tikzcd}$$

$\exists!\ \varphi_n(A)$ s.t $\circledast$ is comm, since $\alpha \varphi_{n-1}(K) \delta_n = \gamma \beta \delta_n = 0$

$\Rightarrow \operatorname{Im}(\varphi_{n-1}(K)\delta_n) \subseteq \ker \alpha = \operatorname{Im} i \cong L_n(F(A))$.

It remains <u>to check</u>:
- $\varphi_n(A)$ does not depend on the choice of P.
- φ_n is a natural transformation.
- φ_n commutes with δ_n. ∎

<u>Right Derived Functors</u>

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor and assume that $\mathcal{A}$ has enough injectives.

Let $A \in \mathcal{A}$ and $A \longrightarrow I^\bullet$ an injective resolution.

<u>Definition</u>: The right derived functors are defined by:

$R^iF(A) := H^n(F(I^\bullet))$.

<u>Theorem</u> With the above hypothesis the following hold.

(1) Each R^nF is an additive functor.

(2) $R^*F = \{R^nF\}_{n \geq 0}$ form a cohomological δ-functor.

(3) R^*F is a universal cohomological δ-functor.

<u>Proof</u>: Consider $F^{op} : \mathcal{A}^{op} \longrightarrow \mathcal{B}^{op}$. F^{op} becomes a right derived functor. Since $\mathcal{A}^{op}$ has enough projectives,

$R^nF(A) := \left(L^nF^{op}\right)^{op}(A) = H^n F(I^\cdot)$ $\blacksquare$

Let A be an R-module, and define:

Definition: $F(-) = \mathrm{Hom}_R(A,-)$; it is a left exact functor, then $\exists$ right derived functor.

$$\mathrm{Ext}^n_R(A,B) := R^n\mathrm{Hom}_R(A,-)\,(B),\ n \geq 0$$

Remark: $\mathrm{Ext}^0_R(A,B) = \mathrm{Hom}(A,B)$, and $\mathrm{Ext}^n_R(A,I) = 0$ $\forall\, n>0$ and I injective.

Derived contravariant functors.

Remark:

Let $F : \mathcal{A} \to \mathcal{B}$ be a contravariant functor.

(1) If F is left exact and $\mathcal{A}$ has enough projectives, then one can define the right derived functors $\{R^nF\}_{n\geq 0}$ such that $R^0F = F$.

(2) If F is right exact and $\mathcal{A}$ has enough injectives, then one can define the left derived functors $\{L_nF\}_{n\geq 0}$ such that $L_0F = F$.

They are universal δ-functors.

(1), ((2) is done similarly). The functor F can be considered as:

$F : \mathcal{A}^{op} \to \mathcal{B}$; it is left exact and $\mathcal{A}^{op}$ has enough injectives $\Rightarrow$ R^iF exists.

Example: Let B be an R-module, and define $F(-) = \mathrm{Hom}_R(-,B)$; it is a covariant left exact functor, therefore $R^i\mathrm{Hom}_R(-,B)$ exists.

Theorem: If A and B are R-modules, then:

$$\mathrm{Ext}^i_R(A,B) = R^i\mathrm{Hom}_R(-,B)\,(A) \cong R^i\mathrm{Hom}_R(A,-)\,(B)\quad \forall\, i \geq 0.$$

Proof: in the next lesson $\blacksquare$

(11) Adjoint functors and left/right exactness.

Theorem

Let $\mathcal{A}$, $\mathcal{B}$ be two abelian categories, and let $(L, R) : \mathcal{A} \longrightarrow \mathcal{B}$ be an adjoint pair of additive functors. Then:

(1) L is right exact. (In particular, if $\mathcal{A}$ has enough projectives, $\exists\, L_n(L)$.)

(2) R is left exact. (——, —— $\mathcal{A}$ —— injectives, $\exists\, R^n(R)$)

Proof: (2) $\Rightarrow$ (1) $(R^{op}, L^{op}) : \mathcal{A}^{op} \longrightarrow \mathcal{B}^{op}$ is also an adjoint pair. By (2) L^{op} is left exact, hence L is right exact.

(2) Let $0 \to B' \to B \to B'' \to 0$ be an exact sequence in $\mathcal{B}$. We have the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{B}}(L(A), B') \arrow[r] \arrow[d, "\wr\ \tau_{AB'}"] & \mathrm{Hom}_{\mathcal{B}}(L(A), B) \arrow[r] \arrow[d, "\wr\ \tau_{AB}"] & \mathrm{Hom}_{\mathcal{B}}(L(A), B'') \arrow[d, "\wr\ \tau_{AB''}"] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A, R(B')) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A, R(B)) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A, R(B''))
\end{tikzcd}$$

The upper row is exact since $\mathrm{Hom}_{\mathcal{B}}(L(A), -)$ is left exact. Next we apply the Yoneda lemma. ∎

Proposition: Let R and S be two rings and let ${}_R B_S$ be an R-S bimodule.

Then:

$$\begin{tikzcd}
\text{mod-}R \arrow[r, bend left, "- \otimes_R B"] \arrow[r, phantom, "\text{adjoint}"] & \text{mod-}S \arrow[l, bend left, "\mathrm{Hom}_S(B, -)"]
\end{tikzcd}$$

In particular, $- \otimes_R B$ is right exact fct.

(Sketch of) Proof: Check:
- $\mathrm{Hom}_S(B, C)$ is a right R-module for all C right S-module
- $A \otimes_R B$ is a right S-module for all A right R-mod.

- $\tau : \mathrm{Hom}_S(A \otimes_R B, C) \xrightarrow{\cong} \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$

$$f \longmapsto (a \longmapsto \tau f(a) : B \longrightarrow C)$$
$$b \to f(a \otimes b)$$

$$\tau^{-1} : \quad a \otimes b \longmapsto g(a)(b) \longleftarrow g \qquad \blacksquare$$

Definition: Let B be an R–S bimodule (left R-module $=$ R–Ab-bimodule). We define:

$\mathrm{Tor}_n^R(A, B) := L_n(- \otimes_R B)(A)$ they are S-modules.

In particular, $\mathrm{Tor}_0^R(A, B) = A \otimes B$ and
$\mathrm{Tor}_n^R(P, B) = 0 \ \forall \ n > 0$ and $\forall P$ projective.
since $\mathrm{Tor}_n^R(A, B) = H_n(P_\bullet \otimes B)$, $P_\bullet \longrightarrow A$ proj resolution.

Theorem: $\mathrm{Tor}_n^R(A, B) \cong L_n(- \otimes_R B)(A) \cong L_n(A \otimes_R -)(B)$, $\forall n \geq 0$.

Proof: in the next lesson. $\blacksquare$