April 25, 2006

Lesson #28 (33) K(A) and D(A) as triangulated categories

<u>Definition</u>: An additive category $\mathcal{K}$ is called <u>triangulated</u> if the following hold:

(1) $\exists\, T : \mathcal{K} \to \mathcal{K}$ an equivalence (translation functor)

(2) $\exists$ class of distinguished triangles (exact triangles) satisfying the axioms:

(TR1) $\forall\, u : A \to B$ $\exists$ an exact triangle (u, v, w) : $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$

- $(id_A, 0, 0)$ is an exact triangle
- $(u, v, w) \cong (u', v', w')$ then (u, v, w) exact $\Leftrightarrow (u', v', w')$ exact.

(TR2) (Rotation) (u, v, w) exact $\Leftrightarrow$ $(v, w, -Tu)$ exact

(TR3) (Morphisms) If (u, v, w) and (u', v', w') are exact triangles

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\forall f"'] & B \arrow[r, "v"] \arrow[d, "\forall g"] & C \arrow[r, "w"] \arrow[d, "\exists h"] & TA \arrow[d, "Tf"] \\
A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & TA'
\end{tikzcd}$$

$\exists\, h$ st we get a morphism of triangles.

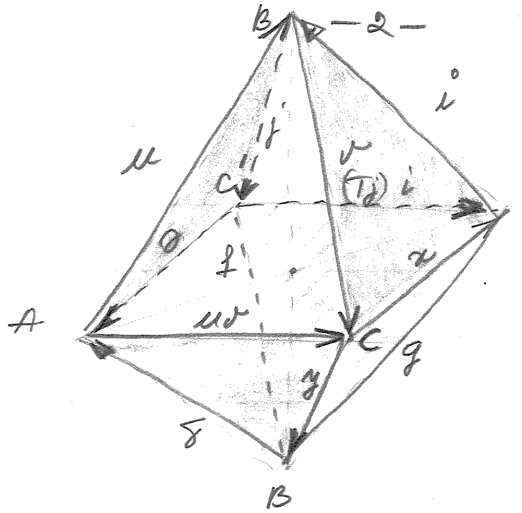
(TR4) (The octahedral axiom)

$$A \xrightarrow{u} B \xrightarrow{j} C' \xrightarrow{\partial} TA$$
$$B \xrightarrow{v} C \xrightarrow{x} A' \xrightarrow{i} TB$$
$$A \xrightarrow{uv} C \xrightarrow{y} B' \xrightarrow{\delta} TA$$

exact triangles

$\exists$ exact triangle $C' \xrightarrow{f} B' \xrightarrow{g} A' \xrightarrow{(Tj)\, i} \Sigma C'$

And the following hold: $\delta f = \partial$, $u\delta = ig$, $x = gy$, $yv = fj'$



Remark :

$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$

exact triangle $\Rightarrow$

$vu = 0, \quad wv = 0$

Pf:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] & B \arrow[r, "v"] \arrow[d, "v"] & C \arrow[r, "w"] \arrow[d, equal] & TA \arrow[d] \\ 0 \arrow[r] & C \arrow[r, equal] & C \arrow[r] & 0 \end{tikzcd}$$

Theorem : $\mathcal{K}(\mathcal{A})$ is a triangulated category.

Proof :

$\mathcal{K}(\mathcal{A})$ is an additive category. (see lesson # 6).

$T(A_\bullet) := A_\bullet[-1] \quad \forall\ A_\bullet \in Ch(\mathcal{A})$ is an equivalence of categories

strict triangles :

$u : A_\bullet \longrightarrow B_\bullet$ morphism in $Ch(\mathcal{A})$

$0 \longrightarrow B_\bullet \xrightarrow{v_\bullet} \mathrm{cone}(u_\bullet) \xrightarrow{\delta} A_\bullet[-1] \longrightarrow 0$

$\mathrm{Cone}(u_\bullet)_n = A_{n-1} \oplus B_n$

$$\begin{tikzcd} A_{n-1} \arrow[r, "-"] \arrow[dr, "-"] & A_{n-2} \\ \oplus & \oplus \\ B_n \arrow[r, "+"'] & B_{n-2} \end{tikzcd}$$

$A_\bullet \xrightarrow{u} B_\bullet \xrightarrow{v_\bullet} \mathrm{Cone}(u_\bullet) \xrightarrow{\delta} A_\bullet[-1]$ is called a strict triangle

Any triangle isomorphic $\boxed{\text{in } \mathbf{K}(\mathcal{A})}$ to a strict triangle is called exact triangle.

up to homotopy equivalence

(TR1): any $u_\bullet$ stays in an exact (strict) triangle. ✓

any triangle isomorphic to an exact triangle is exact ✓.

$u_\bullet = id_{A_\bullet}$

$$\begin{tikzcd} A_\bullet \arrow[r] \arrow[d, equal] & A_\bullet \arrow[r] \arrow[d, "id_A"] & \mathrm{Cone}(id_{A_\bullet}) \arrow[r] \arrow[d, "0\ \text{hom equiv}"] & A_\bullet[-1] \arrow[d, "id_{A_\bullet[-1]}"] \\ A_\bullet \arrow[r] & A_\bullet \arrow[r] & 0 \arrow[r] & A_\bullet[-1] \end{tikzcd}$$

- $\mathrm{Cone}(id_{A_\bullet}) \longrightarrow 0$ homotopy equivalence (Lesson # 5 Exercise)

(TR2): let $A_\bullet \xrightarrow{u_\bullet} B_\bullet \xrightarrow{v} \mathrm{Cone}(u_\bullet) \xrightarrow{\delta} A_\bullet[-1]$ be a strict triangle, we want $B_\bullet \longrightarrow \mathrm{Cone}(u_\bullet) \longrightarrow A_\bullet[-1] \xrightarrow{-u[-1]} B[-1]$ to be isomorphic in $\mathbf{K}(\mathcal{A})$ to a strict triangle.

$$\begin{tikzcd} B_\bullet \arrow[r, "\vartheta"] \arrow[d, equal] \arrow[dr, phantom, "///"] & \mathrm{Cone}(u_\bullet) \arrow[r] \arrow[d, equal] & \mathrm{Cone}(\vartheta_\bullet) \arrow[r] \arrow[d, "\theta_\bullet"] & B_\bullet[-1] \arrow[d, equal] \\ B_\bullet \arrow[r, "\vartheta"] & \mathrm{Cone}(u_\bullet) \arrow[r, "\delta"] & A_\bullet[-1] \arrow[r, "-u_\bullet[-1]"] & B_\bullet[-1] \end{tikzcd}$$

$\exists\, \theta_\bullet$ a homotopy equivalence. (Exercise Hw #5)

(TR3) We may assume that the two triangles are strict.

$$\begin{tikzcd} A_\bullet \arrow[r, "u_\bullet"] \arrow[d, "f_\bullet"] \arrow[dr, phantom, "///"] & B_\bullet \arrow[r] \arrow[d, "g_\bullet"] \arrow[dr, phantom, "///"] & \mathrm{Cone}(u_\bullet) \arrow[r] \arrow[d, "h_\bullet"] \arrow[dr, phantom, "///"] & A_\bullet[-1] \arrow[d] \\ A'_\bullet \arrow[r, "u'_\bullet"] & B'_\bullet \arrow[r] & \mathrm{Cone}(u'_\bullet) \arrow[r] & A'_\bullet[-1] \end{tikzcd}$$

$h(a,b) := (f(a), g(b))$ the diagram is commutative.

(TR4) We may assume that all the triangles are strict.

$A_\bullet \xrightarrow{u_\bullet} B_\bullet \xrightarrow{j} \mathrm{Cone}(u_\bullet) \xrightarrow{\partial} A_\bullet[-1]$ $\qquad C' := \mathrm{Cone}(u_\bullet)$

$B_\bullet \xrightarrow{v_\bullet} C_\bullet \xrightarrow{x} \mathrm{Cone}(v_\bullet) \xrightarrow{\varepsilon} B_\bullet[-1]$ $\qquad A' := \mathrm{Cone}(v_\bullet)$

$A_\bullet \xrightarrow{v_\bullet u_\bullet} C_\bullet \xrightarrow{y} \mathrm{Cone}(v_\bullet u_\bullet) \xrightarrow{\delta} A_\bullet[-1]$ $\qquad B' := \mathrm{Cone}(v_\bullet u_\bullet)$

<u>Want:</u> $C'_\bullet \xrightarrow{f} B'_\bullet \xrightarrow{g} A'_\bullet \xrightarrow{(T_j)\, i} C'_\bullet[-1]$

$$\begin{tikzcd} \mathrm{Cone}(u_\bullet) = C' = \cdots \arrow[r] & A_n \oplus B_{n+1} \arrow[r] \arrow[d, "\mathrm{id}\ \oplus\ v_{n+1}"] & A_{n-1} \oplus B_n \arrow[r] \arrow[d, "\mathrm{id}\ \oplus\ v_{n+1}"] & A_{n-2} \oplus B_{n-2} \arrow[r] & {} \\ \mathrm{Cone}(v_\bullet u_\bullet) = B' = \cdots \arrow[r] & A_n \oplus C_{n+1} \arrow[r] & A_{n-1} \oplus C_n \arrow[r] & \cdots \end{tikzcd}$$

$f_\bullet \downarrow$

$$\begin{tikzcd} (a, b) \arrow[r, mapsto] \arrow[d] & (-d(a),\, d(b) - u_\bullet(a_\bullet)) \arrow[d, "v_\bullet"] \\ (a, v(b)) \arrow[r] & (-d(a),\, d(v(b)) - v_\bullet u_\bullet(a_\bullet)) \end{tikzcd}$$

$dv = vd$

$\boxed{\delta f = \partial}$ ✓ $\qquad \boxed{y v = f j}$ $\qquad y(v(b)) = (0, v(b))$

$f j(b) = f((0, b)) = (0, v(b))$

$$\begin{array}{lcl} g_\bullet : \mathrm{Cone}(\vartheta_\bullet u_\bullet) = B'_\bullet & \to A_n \oplus C_{n+1} \longrightarrow & A_{n-1} \oplus C_n \longrightarrow A_{n-2} \oplus C_{n-1} \longrightarrow \\ \downarrow & u_n \downarrow \quad \downarrow id & \downarrow u_{n-1} \quad \downarrow id \qquad u_{n-2}\downarrow \quad \downarrow id \\ \mathrm{Cone}(\vartheta_\bullet) = C'_\bullet & \to B_n \oplus C_{n+1} \longrightarrow & B_{n-1} \oplus C_n \longrightarrow B_{n-2} \oplus C_{n-1} \longrightarrow \end{array}$$

$$\begin{array}{ccc} (a,c) & \longrightarrow & (-d(a),\ d(c) - \vartheta_\bullet u_\bullet(a)) \\ \downarrow & & \downarrow \\ (u_\bullet(a), c) & \longrightarrow & (-d(u_\bullet(a)),\ d(c_\bullet) - \vartheta u_\bullet(a)) \end{array} \qquad ud = du.$$

$\boxed{u\delta = ig}$ $\quad u\delta(a,c) = u(a)$

$ig(a,c) = i(u(a),c) = u(a).$

$\boxed{x = gy}$ $\quad g(y(c_\bullet)) = (0,c) = x(c).$

$(Tj)i : \mathrm{Cone}(\vartheta_\bullet) \longrightarrow \mathrm{Cone}(u_\bullet)[-1]$

$(b,c) \overset{i}{\longmapsto} (b,0) \overset{T(j)}{\longmapsto} (0,b)$

$$\begin{tikzcd} C' \arrow[r, "f"] \arrow[d, equal] & B' \arrow[r, "g"] \arrow[d, equal] & A' \arrow[r, "(Tj)i"] \arrow[d, "\gamma"] & C'[-1] \arrow[d, equal] \\ C' \arrow[r, "f"'] & B' \arrow[r] & \mathrm{Cone}(f) \arrow[r] & C'[-1] \end{tikzcd}$$

<u>γ is well defined</u> //

$$\begin{array}{lcccc} A' = & \cdots \longrightarrow B_m \oplus C_{n+1} \longrightarrow & B_{n-1} \oplus C_n & \longrightarrow & B_{m-2} \oplus C_{n-1} \longrightarrow \cdots \\ \downarrow \gamma & & (b,c) \mapsto (0,b,0,c) \big\downarrow & & \big\downarrow \\ \mathrm{Cone}(f_\bullet) = & & A_{n-2} \oplus B_{n-1} \oplus A_{n-1} \oplus C_n & \longrightarrow & A_{n-3} \oplus B_{m-2} \oplus A_{n-2} \oplus C_{n-1} \end{array}$$

$$\begin{array}{ccc} (b,c) & \longrightarrow & (-d(b),\ d(c) - \vartheta_\bullet(b)) \\ \downarrow & /\!/ & \downarrow \\ (0,b,0,c) & \longrightarrow & (0,\ d(b),\ (0,\ d(c) - 0,\ \vartheta_\bullet(b))) \end{array}$$

<u>γ is a chain homotopy equivalence</u> //

$$\begin{array}{ccccc} \varphi_\bullet : & \mathrm{Cone}(f_\bullet) = & \longrightarrow & (A_{n-2}\oplus B_{n-1}) \oplus (A_{n-1}\oplus C_n) & \longrightarrow \\ & \downarrow & & \downarrow & \\ & A' = & \longrightarrow & B_{n-1}\oplus C_n & \longrightarrow \end{array}$$

with $u : A_{n-1} \to B_{n-1}$.

$$(a_{n-2}, b, a_{n-1}, c)$$
$$\downarrow$$
$$(b + u(a_{n-1}), c)$$

$\boxed{\varphi\gamma = id_{A'}}$ $\quad \varphi\gamma(b,c) = \varphi(0, b, 0, c) = (b,c)$ ✓

$\boxed{\gamma\varphi \sim id_{\mathrm{Cone}(f_\bullet)}}$ $\quad \gamma\varphi_\bullet(a_{n-2}, b, a_{m-1}, c) = \gamma_\bullet(b + u(a_{n-1}), c)$
$= (0, b + u(a_{n-1}), 0, c)$

a_{n-2}, b, a_{n-1}, c

$$\begin{array}{ccccc} \mathrm{Cone}(f_\bullet) = & \longrightarrow & (A_{n-2}\oplus B_{n-1}) \oplus (A_{n-1}\oplus C_n) & \longrightarrow \\ \downarrow & \swarrow s & \gamma\varphi \downarrow\downarrow id & \swarrow s \\ \mathrm{Cone}(f_\bullet) = & \longrightarrow & (A_{n-2}\oplus B_{n-1}) \oplus (A_{n-1}\oplus C_n) & \longrightarrow \end{array}$$

Exercise*: Find s s.t. $ds + sd = \gamma\varphi - id$.

Definition: Let $\mathcal{K}$ be a triangulated category and $\mathcal{A}$ an abelian category. An additive functor $H : \mathcal{K} \to \mathcal{A}$ is called
(co) homological functor if the following holds
(u, v, w) exact triangle $\Longrightarrow$ $T^{-1}C_\bullet \to A_\bullet \to B_\bullet \to C_\bullet \to TA_\bullet \to TB_\bullet$
$\to TC_\bullet \to T^2A_\bullet \to \cdots$ induces an exact sequence
$\cdots \to H(A_\bullet) \to H(B_\bullet) \to H(C_\bullet) \to H(TA_\bullet) \to H(TB_\bullet) \to$

$H_i := H(T^iA) = H^i(A)$.

Example*: (1) $\mathrm{Hom}_K(X, -)$ is a cohomological functor.
(2) $\{H^0(-)\}$ is a cohomological functor.

Definition: If $H : \mathcal{K} \to \mathcal{A}$ is a (co) homological functor
$S = \{ s \in \mathcal{K} \mid H^i(s) \text{ isom } \forall i \}$ is called system arising from a homological functor. (i.e. $H(T^is)$ isom $\forall i$)

<u>Theorem</u>*: If S is a system arising from a (co)homological functor, then
(1) S is a multiplicative system
(2) $\mathcal{K}$ triangulated category $\Rightarrow$ $S^{-1}\mathcal{K}$ is a triangulated category and $\mathcal{K} \longrightarrow S^{-1}\mathcal{K}$ is a morphism of triangulated categories.

<u>Corollary</u>: $D(A)$ is a triangulated category once Q is locally small.

<u>Definition</u>: A morphism $\mathcal{K} \longrightarrow \mathcal{K}'$ of triangulated categories is an additive functor that commutes with the translation functor T and sends exact triangles to exact triangles.

<u>Proof of the theorem</u>:

(1) • s, t are s.t $H(s), H(t)$ are isomorphisms $\Rightarrow$
$H(st) = H(s)H(t)$ is an isomorphism.

•
$$\begin{tikzcd}
W \arrow[r, "t"] \arrow[d, dotted, "\exists g"'] & X \arrow[r, "uf"] \arrow[d, "f"] & C \arrow[r] \arrow[d, equal, "id"] & W[-1] \arrow[d, "Tg"] \\
Z \arrow[r, "s \in S"'] & Y \arrow[r, "u"'] & C \arrow[r, "\delta"'] & Z[-1]
\end{tikzcd}$$

$H^*(s)$ isomorphism $\Rightarrow$ $H^*(C) = 0$ $\Rightarrow$ $H^*(t)$ isom $\Rightarrow t \in S$

Similarly, we show
$$\begin{tikzcd}
W \arrow[r, "\forall t \in S"] \arrow[d, "\forall g"'] \arrow[dr, phantom, "="] & X \arrow[d, dotted, "\exists f"] \\
Z \arrow[r, dashed, "\exists s \in S"'] & Y
\end{tikzcd}$$

• Assume $sf = sg$, $s \in S$ $\Rightarrow$ $s(f-g) = 0$

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{s} Y'$$

$$\begin{tikzcd}
 & X \arrow[dl, dashed, "\exists h"'] \arrow[d, "f-g"] \arrow[dr, "0"] & & \\
Z \arrow[r, "u"'] & Y \arrow[r, "s"'] & Y' \arrow[r, "\delta"'] & TZ
\end{tikzcd}$$

$H^*(Z) = 0$

$\mathrm{Hom}_{\mathcal{K}}(X,-)$ is a homological functor $\Rightarrow$

$$\mathrm{Hom}_{\mathcal{K}}(X,Z) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(X,Y') \quad \text{is exact}$$

$$f-g \longmapsto s(f-g) = 0$$

$\Rightarrow \exists\, h : X \to Z$ st $uh = f-g$ in $\mathcal{K}$

$\exists \quad X' \xrightarrow{t} X \xrightarrow{h} Z \xrightarrow{w} TX' \qquad H^*(Z) = 0 \Rightarrow t \in \mathcal{S}.$

$(f-g)t = u(ht) = 0. \Rightarrow ft = gt.$

Similarly, $\quad ft = gt \Rightarrow \exists\, s \in S \quad sf = sg$.

(2)

Assume $S^{-1}\mathcal{K}$ exists. We define a translation functor as follows:

$T : S^{-1}\mathcal{K} \longrightarrow S^{-1}\mathcal{K}, \quad T(fs^{-1}) = T(f)T(s)^{-1}$

It is well defined since $s \in S \Rightarrow T(s) \in S$

$s \in S \Rightarrow H^i(s) = H(T^i(s))$ isomorphism $\forall\, i \in \mathbb{Z}$.

<u>Exact triangles on $S^{-1}\mathcal{K}$</u>

$A \xleftarrow{s_1} X \xrightarrow{u} B \qquad us_1^{-1}$

$B \xleftarrow{s_2} Y \xrightarrow{v} C \qquad vs_2^{-1}$

$C \xleftarrow{s_3} C' \xrightarrow{w} T(A) \qquad ws_3^{-1}$

$A \xrightarrow{us_1^{-1}} B \xrightarrow{vs_2^{-1}} C \xrightarrow{ws_3^{-1}} T(A)$ exact triangle if:

$$ws_3^{-1} \rightsquigarrow s_3^{-1}(vs_2^{-1}) = v't_2^{-1} \rightsquigarrow t_2^{-1}(us_1^{-1}) = u't_1^{-1}$$

$$\Rightarrow \boxed{us_1^{-1} = t_2 u' t_1^{-1}} \text{ and } \boxed{vs_2^{-1} = s_3 v' t_2^{-1}}$$

where

$$\begin{tikzcd}
A' \arrow[r, "u'"] \arrow[d, "t_1"] & B' \arrow[r, "v'"] \arrow[d, "t_2"] & C' \arrow[r, "w'"] \arrow[d, "s_3"] & TA' \arrow[d, "Tt_1"] \\
A \arrow[r, dashed, "us_1^{-1}"'] & B \arrow[r, dashed, "vs_2^{-1}"'] & C \arrow[r, dashed, "ws_3^{-1}"'] & TA
\end{tikzcd}$$

is an exact triangle.

<u>Check*</u>: · Exact triangles on $S^{-1}\mathcal{K}$ satisfy the axioms (TR1) – (TR4).

· $\exists$ of a functor of triangulated categories,