

Lesson # 2: ② Abelian categories (cont)

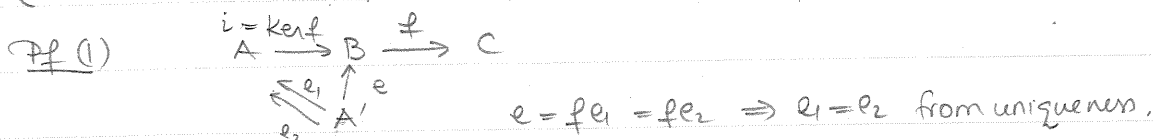
③ Abelian category of chain complexes.

② Abelian categories (cont)

Exercise: Let \mathcal{C} be a category with a 0 object and let $f: B \rightarrow C$

(1) $\ker(f)$ is monic.

(2) $\text{Coker}(f)$ is epi. (prove and show $\text{epi} \not\Rightarrow \text{Coker}(f)$).



Example: i monic $\not\Rightarrow \ker(f) = i$ for some f .

Groups: monic = injective maps.

kernel = monic + image is a normal group.

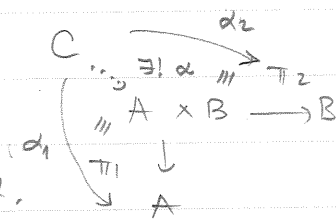
Example: An isomorphism is epi and monic.

Definition: A category \mathcal{A} is called additive if

(1) \mathcal{A} is an Ab -category.

(2) $0 \in \mathcal{A}$

(3) $\forall A, B \in \mathcal{A} \quad \exists A \times B = \text{direct product.}$



Exercise: $A \times B \cong A \oplus B$ in an additive category.

Definition: A category \mathcal{A} is called abelian if.

(1) \mathcal{A} is an additive category.

(2) $\forall f$ monic $f = \ker(\text{Coker}(f))$

Rmk: $\ker = \text{monic}$

(3) $\forall f$ epi $f = \text{Coker}(\ker(f))$.

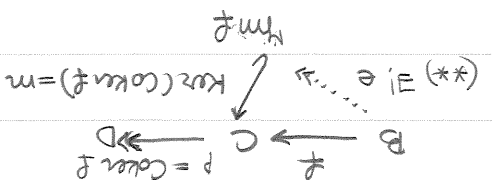
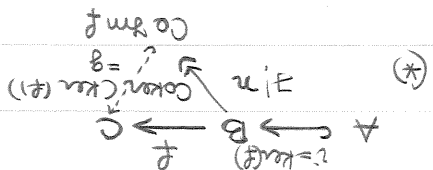
$\text{Coker} = \text{epi}$

Example: $R\text{-mod}$ and $\text{mod-}R$ are abelian categories.

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Theorem: If \mathcal{A} is an abelian category, then $\text{Ch}(\mathcal{A})$ is abelian.

Definition:



Proposition: (1) \exists g monic \wedge f is commutative: $gn = f$

(2) \exists e epi s.t. $(**)$ is commutative: $me = f$

(3) Moreover, $A \xrightarrow{?} B \xrightarrow{f} C \xrightarrow{f} D$

\exists f isomorphism s.t. d.c.



$a=0$
 $m=0$
 $pt=0 \Rightarrow v=0$
 m monic

Proof: (1) and (2) \Rightarrow (3)

$a=0 \Rightarrow \exists$ s.t.

$\tilde{f}n = e \Rightarrow m\tilde{f}n = f$

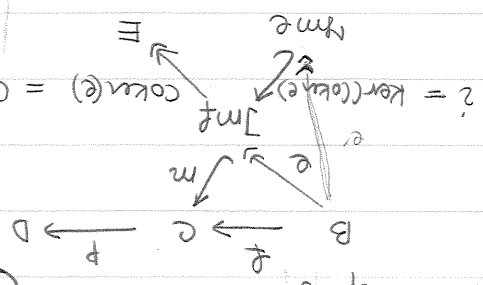
$\tilde{f}n = e$ epi $\Rightarrow \tilde{f}$ epi; $m\tilde{f} = g$ monic $\Rightarrow \tilde{f}$ monic $\Rightarrow \tilde{f}$ isom.

(2) \exists e

$pf = 0$
 $pm = 0$
 $m = \text{ker } p$

$\Rightarrow \exists$ e by universality prop of $\text{ker}(p)$

Exercise: Let \mathcal{A} be an Ab-category
 (1) f monic $\Leftrightarrow \text{ker } f = 0$
 (2) f epi $\Leftrightarrow \text{im } f = 0$



ETS $f = \text{Coker}(m)$, $\text{Im } m = \text{ker } f \Rightarrow \text{Im } m = \text{ker } (\text{Coker } m)$

$= \text{ker } p$

On the other hand $m = \text{ker } p \Rightarrow$ isomorphism \Leftrightarrow epi and mono \Rightarrow

$\text{Coker}(i) = 0$

$\text{Coker}(e) = 0 \Rightarrow e$ epi.

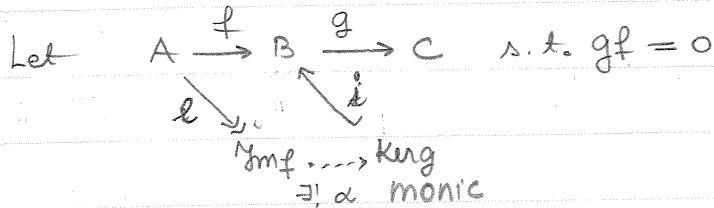
Lemma: $B \xrightarrow{f} C \xrightarrow{f} D$
 $\text{Coker } f = \text{Coker } m$

Exercise: \mathcal{A} abelian category
 $f: B \rightarrow C$ isomorphism $\Leftrightarrow f$ monic and epi

$m = \text{ker } (\text{Coker } f) = \text{ker } (p)$
 $\text{Coker } m = \text{Coker } (\text{ker } p) = p$

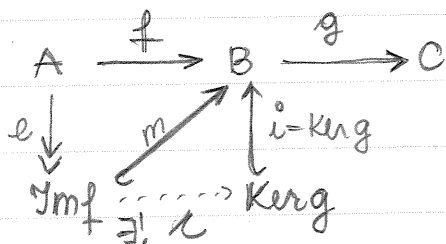
Rmk: Identify the ker map with the object.

Definition-Proposition

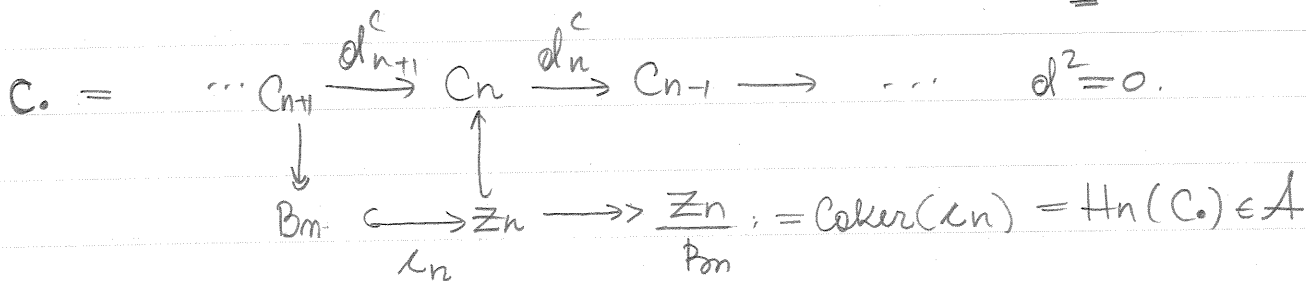


When α is an isomorphism, we call the sequence $A \rightarrow B \rightarrow C$ exact (at B).

Proof:



ETS: $gm = 0$
 $i = \ker g$
 $ia = m$
 m monic $\Rightarrow \alpha$ monic



$H_n: \text{Ch}(A) \rightarrow A$ (covariant) functor
 chain maps, usual way.

H_n is a functor (also on maps)

Definition Let \mathcal{C}, \mathcal{D} any two categories

$F: \mathcal{C} \rightarrow \mathcal{D}$ is called a covariant functor (resp. contravariant)

$C \rightarrow F(C)$

$f: B \rightarrow C \rightsquigarrow F(f): F(B) \rightarrow F(C)$ (resp $F(f): F(C) \rightarrow F(B)$)

if (1) $F(\text{id}_C) = F \text{id}_C$

(2) $F(gf) = F(g)F(f)$ (resp $F(f)F(g)$)

$B \xrightarrow{f} C \xrightarrow{g} D$

$F(B) \xleftarrow{F(f)} F(C) \xleftarrow{F(g)} F(D)$

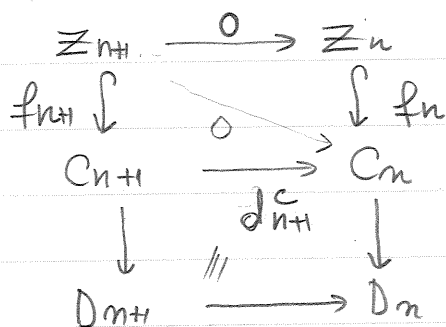
To be done in lesson #3

$\exists \text{Coker}(f_n)$ in $\text{Ch}(A)$.

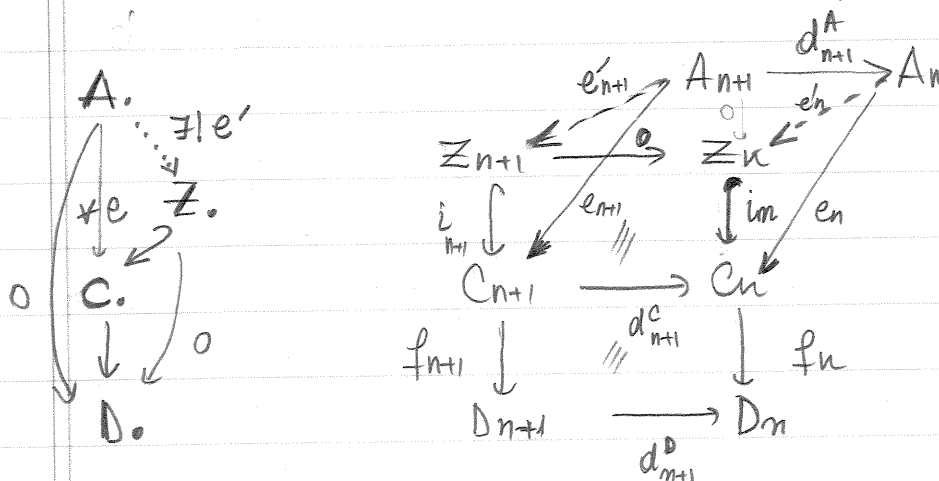
Let $f: C \rightarrow D$. $\exists \text{Ker}(f_n) //$

$\{ \text{Ker}(f_n) \}_{n \in \mathbb{Z}}$

Define $\text{Ker}(f): \mathbb{Z} \rightarrow C$.



$\text{Ker}(f_n)$ sat. the kernel univ. prop. //



$\Rightarrow \exists! e'_n$ s.t. $i_n e'_n = e_n$

Check: $0 e'_{n+1} = e'_n d_{n+1}^A (= 0) //$

$$\begin{aligned}
 i_n e'_n d_{n+1}^A &= e_n d_{n+1}^A = d_{n+1}^C e_{n+1} = d_{n+1}^C \cdot i_{n+1} e'_{n+1} = 0 \\
 \Rightarrow e'_n d_{n+1}^A &= 0.
 \end{aligned}$$

in mono

Let $f: C \rightarrow D$, monic want to show $f = \text{Ker}(\text{Coker } f)$

$\exists \text{TS } f_m$ monic $\forall m \in \mathbb{Z}$, then $f_m = \text{Ker}(\text{Coker } f_m)$

$\Rightarrow f = \text{Ker}(\text{Coker } f)$ by

the above construction.

Ex $\Rightarrow f$ monic $\Leftrightarrow \text{Ker}(f) = 0 \Leftrightarrow \text{Ker}(f_n) = 0 \forall n \in \mathbb{Z}$.

$\Leftrightarrow f_n$ monic $\forall n \in \mathbb{Z}$

Exercise: f epi $\iff f \cdot \text{Coker}(k\alpha f)$.

Therefore, $\text{Coker}(k\alpha f)$ is an abelian category.