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Lesson # 2: ② Abelian categories (cont)

③ Abelian category of chain complexes.

② Abelian categories (cont)

Exercise: Let $\mathcal{C}$ be a category with a 0 object and let $f : B \to C$

(1) $\ker(f)$ is monic.

(2) $\operatorname{Coker}(f)$ is epi. (prove and show epi $\not\Rightarrow$ coker(f)).

Pf (1)
$$\begin{tikzcd} A \arrow[r, "i = \ker f"] & B \arrow[r, "f"] & C \\ & A' \arrow[u, "e"'] \arrow[ul, shift left, "e_1"] \arrow[ul, shift right, "e_2"'] & \end{tikzcd}$$
$e = fe_1 = fe_2 \Rightarrow e_1 = e_2$ from uniqueness.

Example: i monic $\not\Rightarrow$ $\ker(f) = i$ for some f.

Groups: monic = injective maps.

kernel = monic + image is a normal group.

Example: An isomorphism is epi and monic.

Definition: A category $\mathcal{A}$ is called additive if

(1) $\mathcal{A}$ is an Ab-category

(2) $0 \in \mathcal{A}$

(3) $\forall\, A, B \in \mathcal{A}$ $\exists$ $A \times B$ = direct product.

$$\begin{tikzcd} C \arrow[dr, dotted, "\exists!\,\alpha"] \arrow[drr, bend left, "\alpha_2"] \arrow[ddr, bend right, "\alpha_1"'] & & \\ & A \times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \\ & A & \end{tikzcd}$$

Exercise: $A \times B \cong A \oplus B$ in an additive category

Definition: A category $\mathcal{A}$ is called abelian if.

(1) $\mathcal{A}$ is an additive category.

(2) $\forall$ f monic $f = \operatorname{Ker}(\operatorname{Coker}(f))$ Rmk: ker = monic

(3) $\forall$ f epi $f = \operatorname{Coker}(\ker(f))$. Coker = epi

Example: R-mod and mod-R are abelian categories.

3 **Theorem**: If $\mathcal{A}$ is an abelian category, then $Ch(\mathcal{A})$ is abelian.

Definition:

$$(*)\quad \begin{tikzcd} A \arrow[r, hook, "i=\ker(f)"] & B \arrow[r, "f"] \arrow[d, "\exists! n"'] & C \\ & \mathrm{Co}\,\mathrm{Im} f \arrow[ur, dashed, "\mathrm{Coker}(\ker(f)) = g"'] & \end{tikzcd}$$

$$(**)\quad \begin{tikzcd} B \arrow[r, "f"] \arrow[dr, dotted, two heads, "\exists! e"'] & C \arrow[r, two heads, "p=\mathrm{Coker} f"] & D \\ & \mathrm{Im} f \arrow[u, "\ker(\mathrm{Coker} f)=m"'] & \end{tikzcd}$$

Proposition: (1) $\exists!\ g$ monic s.t $(*)$ is commutative: $gn = f$

(2) $\exists!\ e$ epi s.t $(**)$ is commutative: $me = f$

(3) Moreover,

$$\begin{tikzcd} A \arrow[r, hook, "i"] & B \arrow[r, "f"] \arrow[d, "n"'] \arrow[dr, dotted, "e"] & C \arrow[r, two heads, "p"] & D \\ & \mathrm{Co}\,\mathrm{Im} f \arrow[r, dashed, "\tilde{f}"'] \arrow[ur, dashed, "g"] & \mathrm{Im} f \arrow[u, "m"'] & \end{tikzcd}$$

$\exists!\ \tilde{f}$ isomorphism s.t D.C

$ei = 0$

$\left.\begin{array}{l} mei = 0 \\ fi = 0 \\ m \text{ monic}\end{array}\right\} \Rightarrow ei = 0.$

Proof: (1) and (2) $\Rightarrow$ (3)

$\exists!\ \tilde{f}$

$\left.\begin{array}{l} ei = 0 \\ n = \mathrm{Coker}\, i \end{array}\right\} \Rightarrow \exists!\ \tilde{f}$ s.t $\tilde{f}n = e \Rightarrow m\tilde{f}n = f$.

$\tilde{f}n = e$ epi $\Rightarrow \tilde{f}$ epi ; $m\tilde{f} = g$ monic $\Rightarrow \tilde{f}$ monic $\overset{Ex}{\Longrightarrow} \tilde{f}$ isom.

(2) $\exists!\ e$

$\left.\begin{array}{l} pf = 0 \\ pm = 0 \\ m = \ker p \end{array}\right\} \Rightarrow \exists!\ e$ by universality prop of $\ker(p)$

Exercise: Let $\mathcal{A}$ be an Ab-category
(1) f monic $\Leftrightarrow \ker f = 0$.
(2) f epi $\Leftrightarrow \mathrm{Im} f = 0$

e is epi

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "e"] \arrow[dd, "e'"'] & C \arrow[r, "p"] & D \\ & \mathrm{Im} f \arrow[u, "m"'] \arrow[dr, "\mathrm{Coker}(e)"] & \\ \mathrm{Im}\, e \arrow[ur, "i = \ker(\mathrm{Coker}\, e)"] & & E \end{tikzcd}$$

$\mathrm{Coker}(e) = \mathrm{Coker}(i)$

ETS $p = \mathrm{Coker}(mi)$. Indeed, mi monic $\Rightarrow mi = \ker(\mathrm{Coker}\, mi)$
$= \ker p$

On the other hand $m = \ker p \Rightarrow i$ isomorphism $\Leftrightarrow$ epi and mono $\Rightarrow$

$\mathrm{Coker}(i) = 0$
$\parallel$
$\mathrm{Coker}(e) = 0 \Rightarrow e$ epi.

Exercise: $\mathcal{A}$ abelian category
$f: B \to C$ isomorphism $\Leftrightarrow$
f monic and epi

Lemma: $\begin{tikzcd} B \arrow[r, "f"] & C \arrow[r, "\mathrm{Coker} f = \mathrm{Coker}\, m"] & D \\ & \mathrm{Im} f \arrow[u, "m"] & \end{tikzcd}$

$m = \ker(\mathrm{Coker} f) = \ker(p)$

$\mathrm{Coker}\, m = \mathrm{Coker}(\ker p) = p$.
p epi

Rmk: Identify the ker map with the object.

Definition - Proposition

Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, two heads, "e"'] & B \arrow[r, "g"] & C \\ \mathrm{Im} f \arrow[r, dashed, "\exists!\,\alpha"'] & \mathrm{Ker}\, g \arrow[u, "i"'] \end{tikzcd} \quad \text{s.t. } gf = 0$$

monic

When α is an isomorphism, we call the sequence $A \to B \to C$ exact (at B).

Proof:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "e"'] & B \arrow[r, "g"] & C \\ \mathrm{Im} f \arrow[ur, "m"] \arrow[r, dashed, "\exists!\,\alpha"'] & \mathrm{Ker}\, g \arrow[u, "i = \mathrm{Ker}\, g"'] \end{tikzcd}$$

ETS: $\left.\begin{matrix} gm = 0 \\ i = \mathrm{Ker}\, g \end{matrix}\right\} \Rightarrow \exists!\, \alpha$ s.t. $i\alpha = m$

m monic $\underset{Ex}{\Rightarrow}$ α monic ■

$$\begin{tikzcd} C_\bullet = & \cdots C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d] & C_n \arrow[r, "d^C_n"] & C_{n-1} \arrow[r] & \cdots \\ & B_n \arrow[r, hook, "\iota_n"'] & Z_n \arrow[u] \arrow[r, two heads] & \dfrac{Z_n}{B_n} := \mathrm{Coker}(\iota_n) = H_n(C_\bullet) \in \mathcal{A} \end{tikzcd} \qquad d^2 = 0.$$

$H_n : Ch(\mathcal{A}) \longrightarrow \mathcal{A}$ (covariant) functor

chain maps, usual way.

H_n is a functor (also on maps)

To be done in lesson #3

Definition Let $\mathcal{C}, \mathcal{D}$ be any two categories

$F: \mathcal{C} \to \mathcal{D}$ is called a covariant functor (resp. contravariant)

$C \longrightarrow F(C)$

$f: B \to C \rightsquigarrow F(f): F(B) \longrightarrow F(C)$ (resp $F(f): F(C) \to F(B)$)

if (1) $F(id_C) = Fid_C$

(2) $F(gf) = F(g)F(f)$ $\left(\overset{resp}{=} F(f)F(g)\right)$

$B \xrightarrow{f} C \xrightarrow{g} D \qquad F(B) \xleftarrow{F(f)} F(C) \xleftarrow{F(g)} F(D)$

Examples: (1) $id_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$

(2) $M \otimes_R - : R\text{-mod} \to Ab$, $\forall M \in \text{mod-}R$

(3) $Hom_R(M,-) : \text{mod-}R \to Ab$ $\forall M \in \text{mod-}R$ cov

$Hom_R(-,M) : \text{mod-}R - Ab$ contravar.

(4) Forgetfull functor : $\mathcal{U} : Ab \to Sets$ or

$\mathcal{U} : \text{mod-}R \to Sets$

in lesson #3

Theorem: If A is an abelian category, $ch(A)$ is an abelian category.

Proof: Replacing mod-R by A we get $Ch(A)$ is an Ab-category.

$\exists 0 \in A \Rightarrow \exists 0_\bullet$ in $Ch(A)$.

$$P_\bullet = C_\bullet \times D_\bullet := C_\bullet \oplus D_\bullet$$

$$\begin{tikzcd}
P_\bullet \arrow[d, "\pi^C"] & \cdots \arrow[r] & C_{n+1} \times D_{n+1} \arrow[r, "d^P_{n+1}"] \arrow[d, "\pi^C_{n+1}"] & C_n \times D_n \arrow[r, "d^P_n"] \arrow[d, "\pi^C_n"] & C_{n-1} \times D_{n-1} \arrow[r, "d^P_{n-1}"] \arrow[d] & \cdots \\
C_\bullet & \cdots \arrow[r] & C_{n+1} \arrow[r, "d^C_{n+1}"] & C_n \arrow[r] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Check if: $\pi^C_n d^P_{n+1} = d^C_{n+1} \pi^C_{n+1}$ $\exists! \; d^P_{n+1}$

yes, by construction.

$$\begin{tikzcd}
C_{n+1} \times D_{n+1} \arrow[d, dashed, "\exists! \, d^P_{n+1}"] \arrow[dr, bend left, "d^D_{n+1} \circ \pi^D_{n+1}"] \arrow[dd, bend right, "d^C_{n+1} \circ \pi^C_{n+1}"'] & \\
C_n \times D_n \arrow[r, "\pi^D_n"] \arrow[d, "\pi^C_n"] & D_n \\
C_n &
\end{tikzcd}$$

Check if: $(d^P)^2 = 0$

or $d^P_n \circ d^P_{n+1} = 0 \quad \forall n \in \mathbb{Z}$

$$(\pi^C_{m-1} d^P_n) d^P_{n+1} = d^C_n \pi^C_n d^P_{m+1} = d^C_n d^C_{n+1} \pi^C_n = 0.$$

similarly $\pi^D_{m-1} \circ (d^P_n d^P_{n+1}) = 0$

by uniqueness $\Longrightarrow d^P_n d^P_{n+1} = 0$.

Therefore, $Ch(A)$ is an additive category.

Exercise
$\exists\ \mathrm{Coker}(f_\bullet)$ in $\mathrm{Ch}(\mathcal{A})$.

Let $f: C_\bullet \longrightarrow D_\bullet$. $\quad \underline{\exists\ \mathrm{Ker}(f_\bullet)}$ //

Define $\mathrm{Ker}(f_\bullet): Z_\bullet \xrightarrow{\{\mathrm{Ker} f_n\}_{n\in\mathbb{Z}}} C_\bullet$

$$\begin{tikzcd}
Z_{n+1} \arrow[r, "0"] \arrow[d, hook, "f_{n+1}"'] \arrow[dr, "0"] & Z_n \arrow[d, hook, "f_n"] \\
C_{n+1} \arrow[r, "d^C_{n+1}"'] \arrow[d] \arrow[dr, phantom, "/\!/\!/"] & C_n \arrow[d] \\
D_{n+1} \arrow[r] & D_n
\end{tikzcd}$$

$\mathrm{Ker}(f_\bullet)$ sat. the kernel univ. prop. //

$$\begin{tikzcd}
A_\bullet \arrow[dr, dashed, "\exists!\, e'"] \arrow[dd, "\forall e"'] \arrow[ddd, bend right, "0"'] & \\
& Z_\bullet \arrow[dl] \arrow[ddl, bend left, "0"] \\
C_\bullet \arrow[d] & \\
D_\bullet &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_{n+1} \arrow[rr, "d^A_{n+1}"] \arrow[dl, dashed, "e'_{n+1}"'] \arrow[ddl, "e_{n+1}"] \arrow[dr, "0"] & & A_n \arrow[dl, dashed, "e'_n"'] \arrow[ddl, "e_n"] \\
Z_{n+1} \arrow[rr, "0"] \arrow[d, hook, "i_{n+1}"'] & & Z_n \arrow[d, hook, "i_n"] & \\
C_{n+1} \arrow[rr, "d^C_{n+1}"'] \arrow[d, "f_{n+1}"'] & & C_n \arrow[d, "f_n"] & \\
D_{n+1} \arrow[rr, "d^D_{n+1}"'] & & D_n &
\end{tikzcd}$$

$\Rightarrow\ \exists!\ e'_n$ s.t. $i_n e'_n = e_n$

Check: $0\, e'_{n+1} = e'_n\, d^A_{n+1}\ \underline{(=0)}$ //

$$i_n e'_n d^A_{n+1} = e_n d^A_{n+1} = d^C_{n+1} e_{n+1} = d^C_{n+1} \cdot i_{n+1} e'_{n+1} = 0$$

$$\Longrightarrow \quad e'_n d^A_{n+1} = 0.$$

i_n monic

Let $f_\bullet\ C_\bullet \longrightarrow D_\bullet$ monic want to show $f_\bullet = \mathrm{Ker}(\mathrm{Coker}\, f_\bullet)$
ETS f_n monic $\forall\ n \in \mathbb{Z}$, then $f_n = \mathrm{Ker}((\mathrm{Coker}\, f_n))$
$\Longrightarrow f_\bullet = \mathrm{Ker}(\mathrm{Coker}\, f_\bullet)$ by the above construction.

Ex $\Rightarrow$ f monic $\iff \mathrm{Ker}(f) = 0 \iff \mathrm{Ker}(f_n) = 0\ \forall\ n \in \mathbb{Z}$.
$\iff f_n$ monic $\forall\ n \in \mathbb{Z}$

Exercise: f epi $\iff f = \mathrm{Coker}(\ker f)$.

Therefore, $\mathrm{Ch}(\mathcal{A})$ is an abelian category. ▨