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Lesson #19 (20) Auslander-Buchsbaum thm – cont

(21) Auslander–Buchsbaum–Serre thm.

(20) – cont ADDENDUM – Gorenstein dimension, Stable Ext.

Theorem: Let (R, m, k) be a commutative, noetherian local ring. and let A be f.g -R-module s.t $pd_R(A) < \infty$. Then

$$\boxed{pd_R(A) = depth(R) - depth(A)}$$

Proof: (cont) $\boxed{depth(R) \geq 1}$ and $\boxed{depth\, A \geq 1}$

$Hom_R(k, A \oplus R) \cong Hom_R(k, A) \oplus Hom_R(k, R) = 0 \Rightarrow$
$depth\,(A \oplus R) \geq 1 \Rightarrow \exists\ x$ that is $A \oplus R$-regular.
In particular, it is A-regular and R-regular.

$$\left.\begin{aligned} \Rightarrow\ & depth(A/xA) = depth(A) - 1 \\ & depth\,(R/xR) = depth(R) - 1 \\ & pd_{R/xR}(A/xA) = pd_R(A) \end{aligned}\right\} \text{By induction we get the AB equality.}$$

∎

Corollary: $depth_R(k) = 0$ since $m \cdot k = 0$

If $pd_R(k) < \infty$, then $pd_R(k) = depth(R) - depth(k)$
$= depth(R)$

Hence, $\boxed{pd_R(k) = gldim(R) = depth(R)}$, when $pd_R(k) < \infty$.

Our aim in the next section is to characterize the local rings with $pd_R(k) < \infty$ (i.e $gldim(R) < \infty$).

(21)

More invariants of a local ring (R, m, k)

Definitions:

- $\dim(R) = \sup\{d \mid p_0 \subsetneq p_1 \subsetneq \dots \subsetneq p_d \text{ primes ideals in } R\}$
  is called the Krull-dimension of R

  $\operatorname{edim}(R) = \dim_k(m/m^2)$ = minimal number of generators of m.
  is called the embedding dimension of R.

Proposition*: There are inequalities: $\boxed{\operatorname{depth}(R) \le \dim(R) \le \operatorname{edim}(R)}$

Proof: It will not be given in this class.

Definition: The ring R is called regular, if

$$\boxed{\dim(R) = \operatorname{edim}(R)}$$

Definition: The ring R is called Cohen-Macaulay if

$$\boxed{\operatorname{depth}(R) = \dim(R)}$$

Examples:
(1) $\dim(R) = 0$ $\quad$ R regular $\iff$ k is a field.
(2) $R = k[[x_1, \dots, x_n]]$ regular, $\dim(R) = n$
OR
$R = k[x_1, \dots x_n]_{(x_1, \dots, x_n)}$

Examples:
(1) $\dim(R) = 0 \Rightarrow R$ is CM $\quad \dfrac{k[x,y]}{(x^2, y^2)} = k[x,y]$.
(2) $R = \dfrac{k[x,y]}{(xy, y^2)}$
$\dim(R) = 1$ not CM, $\operatorname{depth}(R) = 0$.

(3) $R = k[[x^2, x^3]] \subseteq k[[x]]$
CM of $\dim(R) = 1$, not regular.

<u>Exercise:</u> If R is a regular local ring of dimension d, and $x_1, \dots, x_d$ is a minimum set of generators of m, then $R/(x_1, \dots, x_i)R$ is regular of dimension $(d-i)$.

<u>Proposition*</u>: If R is regular local ring, then R is Cohen-Macaulay. Moreover, any minimal set of generators of m is a regular sequence.

<u>Proof</u>: read p. 106.

<u>Corollary</u>: R regular $\Longrightarrow$ $\mathrm{depth}(R) = \dim(R) = \mathrm{edim}(R)$.

<u>Recall</u>: $\mathrm{gldim}(R) = \mathrm{pd}_R(k)$.

<u>Theorem:</u> The ring R is regular $\Longleftrightarrow$ $\mathrm{gldim}(R) < \infty$.

In this case:

$$\mathrm{gldim}(R) = \mathrm{pd}_R(k) = \mathrm{depth}(R) = \dim(R) = \mathrm{edim}(R).$$

<u>Proof</u>: "$\Longrightarrow$" Assume R is regular. Set $d = \dim(R)$. We make ind on d.

$\boxed{d=0}$ R regular $\Rightarrow$ $R = k$ is a field $\Rightarrow \mathrm{pd}_R(k) = 0$.

$\boxed{(d-1) \Rightarrow d}$, $d \geq 1$.

Let $x_1, \dots, x_d$ be a minimal set of generators of m.

By proposition, it is a regular sequence as well.

Exercise $\Rightarrow$ R/x_1R is also regular $\Longrightarrow$

$$\dim(R/x_1R) = \mathrm{edim}(R/x_1R) = d-1 \Longrightarrow \mathrm{gldim}(R/x_1R) = d-1$$

$$\Longrightarrow \mathrm{gldim}(R) = \mathrm{gldim}(R/x_1R) + 1 = d.$$

Previous prop

since $\mathrm{gldim}(R/x_1R) < \infty$

"$\Longleftarrow$". Assume $\mathrm{gldim}\, R < \infty$. Set $d = \mathrm{gldim}(R)$. We make ind. on d

$\boxed{d=0}$ $\mathrm{gldim}\, R = 0 \iff \mathrm{pd}_R(k) = 0 \iff k$ free R-module $\iff$ R is a field.

$\boxed{(d-1) \Rightarrow d}$, $d \geq 1$.

$\mathrm{gldim}(R) = \mathrm{pd}_R(k) = \mathrm{depth}(R) = d \geq 1 \Rightarrow \exists\, x$ a non-zero divizor on R.

Exercise*: We may choose $x \in m \setminus m^2$.

Claim: $\mathrm{gldim}\,(R/xR) = d-1$, where $(R/xR, m/xR, k)$ is a local r

It is enough to show that $\mathrm{gldim}(R/xR) < \infty$, and use $\mathrm{gldim}(R) = d$

Pf: Since $\mathrm{pd}_R(k) \neq 0$, the exact sequences of R-modules

$$0 \to m \to R \to k \to 0 \quad \text{implies:}$$

$\mathrm{pd}_R(m) = d-1$

x is also an m-regular element, then by the third change of rings theorem, we obtain:

(+) $\mathrm{pd}_{R/xR}(m/xm) = \mathrm{pd}_R(m) = d-1$

The following is ~~are~~ exact sequence of R/xR-modules:

$$0 \to xR/xm \to m/xm \to m/xR \to 0$$

with $0 \neq \hat{x} \in xR/xm$, $\hat{x} \leftarrow 1$, $xR/xm \cong k$.

We show that this sequence splits: Assume $(x, x_2, \dots, x_r) = m$ where $r = \mathrm{edim}(R)$. (Here we use $x \in m \setminus m^2$).

Then
$$\frac{m}{xm} = \frac{(x_2, \dots x_r)R + xR}{xm} = \frac{\overbrace{(x_2, \dots, x_r)R + xm}^{I}}{xm} + \frac{xR}{xm}$$

Let $\hat{\alpha} \in \frac{I}{xm} \cap \frac{xR}{xm} \Rightarrow \exists\ x\hat{a} = x_2\hat{a}_1 + \dots + x_r\hat{a}_r$

$\Rightarrow xa = x_2a_1 + \dots + x_ra_r + xb, \quad b \in m$

$\Rightarrow x(a-b) = x_2a_1 + \dots + x_ra_r$

If $a \notin m \Rightarrow a-b \in \mathcal{U}(R) \Rightarrow x \in (x_2, \dots, x_r)$ $\times$

Hence, $a \in m \Rightarrow \hat{\alpha} = 0 \Rightarrow \frac{I}{xm} \cap \frac{xR}{xm} = 0$

Therefore, $m/_{xm} \cong k \oplus m/_{xR}$

$\Rightarrow \operatorname{pd}_{R/xR}(k) \le \operatorname{pd}_{R/xR}\left(m/_{xm}\right) = d-1$ $\boxed{\text{claim}}$

By induction, R/xR is a regular ring of dimension $d-1$.

<u>Claim:</u> R/xR regular, $x \in m \setminus m^2$ regular element $\Rightarrow$ R regular.

Pf. Let $\bar{x}_2, \dots, \bar{x}_d$ minimal set of generators of $m/_{xR}$.

Then $(x, x_2, \dots, x_d)$ is a minimal set of generators of m.

(here we use $x \notin m^2$). $\Rightarrow \operatorname{edim}(R) = d$.

Since $d = \operatorname{gldim}(R) = \operatorname{pd}_R(k) = \operatorname{depth}(R) \le \dim(R) \le \operatorname{edim}(R) = d$

$\Rightarrow$ equality everywhere so R is regular of dimension d. ∎

<u>Corollary</u>: If R is regular, then R_p is regular $\forall\ p \in \operatorname{Spec}(R)$

<u>Proof</u>: R regular $\Rightarrow \operatorname{pd}_R(k) < \infty \Rightarrow \exists\ P_\cdot \to k$ a finite projective resolution of k. Localization is an exact functor

$\Rightarrow \bar{S}^{-1}P_\cdot \to \bar{S}^{-1}k$ is also a proj resolution.

In particular, $\underset{R_p}{pd}(k(p)) < \infty \;\; \forall \, p \in Spec(R)$, where $k(p) = \frac{R_p}{pR_p}$.

[redacted]

<u>Definition</u>: A local ring R is called <u>Gorenstein</u> **if $id(R) < \infty$.**

<u>Example</u>: $k[x_1, \dots x_n] / (x_1^2 - x_2^2, \dots, x_1^2 - x_n^2, x_1x_2, x_1x_3, \dots, x_{n-1}x_n)$.

<u>Proposition</u>*: For A a f.g R module we have

$$\underset{R}{id}(A) = \sup\{ i \mid Ext_R^i(k, A) \neq 0 \}$$

<u>Corollary</u>*: If R is Gorenstein, then R is CM and

$$\boxed{id(R) = depth(R) = dim(R)}$$

<u>Example</u>: $R = \frac{k[X,Y]}{(X^2, XY, Y^2)}$ is CM but not Gorenstein.

<u>Corollary</u>: If R is regular, then R is Gorenstein.

<u>Proof</u>: R regular $\Longrightarrow pd_R(k) < \infty \overset{Ext\,def}{\Longrightarrow} Ext_R^i(k, R) = 0 \;\forall\, i \gg 0$
$\underset{Prop^*}{\Longrightarrow} id(R) < \infty$.

<u>Remark</u>: $\underset{R}{pd}(k) < \infty \iff Ext_R^i(k, k) = 0$ for some i.

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<u>ADDENDUM</u>:
- Gorenstein dimension
- Unstable Ext and Stable Ext

<u>Definition</u>: Let (R, m, k) be a local commutative noetherian ring and let A be a finitely generated R-module.

$$A^* = Hom_R(A, R) \;; \qquad A \xrightarrow{\partial_A} A^{**} = Hom_R(Hom(A, R), R)$$
$$a \longmapsto [f \longmapsto f(a)]$$

A is called <u>reflexive</u>, if ∂_A is an isomorphism

Definition (Auslander–Bridger):
We say that A has Gorenstein dimension ~~zero~~, denoted by $\mathrm{Gdim}_R A = 0$ if the following hold:

(1) A is reflexive.
(2) $\mathrm{Ext}^i_R(A,R) = 0 \;\forall\; i \geq 1$
(3) $\mathrm{Ext}^i_R(A^*,R) = 0 \;\forall\; i \geq 1$

$$\mathrm{Gdim}_R(A) = \sup\{ d \mid \exists\; 0 \to G_d \to G_{d-1} \to \dots \to G_0 \to A \to 0\}$$
$$\mathrm{Gdim}_R(G_i) = 0 \quad \forall \quad 0 \leq i \leq d.$$

Remark:
$\mathrm{Gdim}_R(A)$ has properties similar to the $\mathrm{pd}_R(A)$. In particular,
$\mathrm{Gdim}_R(A) < \infty \Longrightarrow \mathrm{Gdim}(A) = \mathrm{depth}\, R - \mathrm{depth}\, A$.
Remark: It is not known that the General change of rings theorem holds for Gdim.

Theorem (Auslander–Bridger): The following are equivalent:
(i) R is Gorenstein.
(ii) $\mathrm{Gdim}_R(k) < \infty$
(iii) $\mathrm{Gdim}_R(A) < \infty \;\forall\; A$ f.g. R-module.

Some of my research
- Define: $G^*\mathrm{dim}$; construct examples of A, $\mathrm{Gdim}\, A < \infty$.
- Define: Gdim (Gpd) for complexes.
- Independence of the conditions (1), (2), (3) in the definition of Gdim 0.

- Theorem (—, Avramov)

(1) R Gorenstein $\Longleftrightarrow$ $\mathrm{rank}_k\, \widehat{\mathrm{Ext}}^i_R(k,k) < \infty$ for some (all) $i \in \mathbb{Z}$.
(2) R regular $\Longleftrightarrow$ $\widehat{\mathrm{Ext}}^i_R(k,k) = 0$ for some (all) $i \in \mathbb{Z}$.

Definition: Let $P_\bullet \to k$ projective resolution ($k \to I^\bullet$ inj resolution)

$$\mathrm{Ext}^i_R(k,k) = H^i(\mathrm{Hom}(P_\bullet, k)) \cong H^i(\mathrm{Hom}(k, I^\bullet))$$

$$\cong H^i(\mathrm{Hom}(P_\bullet, P_\bullet))$$

Recall: $\mathrm{Hom}(P_\bullet, P_\bullet)_n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}(P_i, P_{i+n})$

$$\cup\!\mid$$

unstable Hom: $\overline{\mathrm{Hom}}(P_\bullet, P_\bullet)_n = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(P_i, P_{i+n})$

stable Hom: $\widehat{\mathrm{Hom}}_R(P_\bullet, P_\bullet)_n = \dfrac{\mathrm{Hom}(P_\bullet, P_\bullet)}{\overline{\mathrm{Hom}}(P_\bullet, P_\bullet)}$

$$\widehat{\mathrm{Ext}}^i_R(k,k) \stackrel{\mathrm{def}}{=} H^i(\widehat{\mathrm{Hom}}_R(P_\bullet, P_\bullet)) \qquad \forall\, i \in \mathbb{Z}.$$