 Tuesday, January 10, 2006

# Lesson #1 : ① Complexes of R modules, Abelian Categories

## ① Complexes of R-modules

$R$ = associative ring

mod-$R$ = the category of right $R$-modules

$R$-mod = the category of left $R$-modules

Notation: $A, B, C, D, \ldots \in$ mod-$R$.

$f : A \to B$ $R$-module homomorphism.

$\mathrm{Ker}(f)$, $\mathrm{Im}(f)$, $\mathrm{Coker}(f)$

Definition: $A \xrightarrow[f]{} B \xrightarrow[g]{} C$ is called exact sequence if $\mathrm{Im} f = \ker g$.

Examples: (1) $\mathbb{Z} \xrightarrow[\binom{2}{-1}]{f} \mathbb{Z}^2 \xrightarrow[(1\ 2)\cdot]{g} \mathbb{Z}$ exact

$a \longmapsto \begin{pmatrix} 2a \\ -a \end{pmatrix}$ $\quad (1\ 2)\begin{pmatrix} a \\ b \end{pmatrix} = a + 2b$

(2) $\mathbb{Z} \xrightarrow[\binom{4}{-2}]{f} \mathbb{Z}^2 \xrightarrow[(1\ 2)]{g} \mathbb{Z}$ not-exact.

$a \longrightarrow \begin{pmatrix} 4a \\ -2a \end{pmatrix} \longmapsto 4a - 4a = 0$

But $\mathrm{Im} f \subsetneq \ker f$.

Definition: $C_\bullet = \cdots \to C_{n+1} \xrightarrow{d^C_{n+1}} C_n \xrightarrow{d^C_n} C_{n-1} \xrightarrow{d^C_{n-1}} \cdots$ (notation)

A chain complex of $R$-modules $C_\bullet$ is a fam $\{C_n\}_{n\in\mathbb{Z}}$ of $R$-modules and a fam $\{d^C_n\}_n$ of ring hom $\forall n \in \mathbb{Z}$ s.t $d^C_m \circ d^C_{m+1} = 0$

Notation: $C = C_\bullet$

$d = d^C = \{d^C_n\}_{n\in\mathbb{Z}}$. $\quad d \circ d = d^2 = 0$.

differentials

Examples: (1) Any module can be viewed as an chain complex.

(2) $G_\bullet = \bigoplus_{n\in\mathbb{Z}} G_n$ is a graded $R$-module, $d = 0$

$\Rightarrow (G_\bullet, d_\bullet)$ is a chain complex $\cdots \to G_{n+1} \xrightarrow{0} G_n \xrightarrow{0} G_{n-1} \to \cdots$

To each Complex $C_\bullet$ corresponds 3 graded modules:

cycles: $Z_\bullet(C_\bullet) = \{Z_n(C_\bullet)\}_{n\in\mathbb{Z}}$ $\quad Z_n(C_\bullet) = \operatorname{Ker}(d_n^c) \subseteq C_n$

bounderies: $B_\bullet(C_\bullet) = \{B_n(C_\bullet)\}_{n\in\mathbb{Z}}$ $\quad B_n(C_\bullet) = \operatorname{Im}(d_{n+1}^c) \subseteq C_n$

homology: $H_\bullet(C_\bullet) = \left\{\dfrac{Z_n(C_\bullet)}{B_n(C_\bullet)}\right\}_{n\in\mathbb{Z}}$ $\quad d^2=0 \Rightarrow B_n(C_\bullet) \subseteq Z_n(C) \subseteq C_n \ \forall n\in\mathbb{Z}$

Definition: $\begin{cases} C_\bullet \text{ is acyclic} \Leftrightarrow H_n(C) = 0 \ \forall n\in\mathbb{Z} & \left(\xrightarrow{\cdot 2} \mathbb{Z}_4 \xrightarrow{\cdot 2} \mathbb{Z}_4 \xrightarrow{\cdot 2}\right) \\ C_\bullet \text{ is exact at } n \text{ if } B_n(C_\bullet) = Z_n(C_\bullet) \Leftrightarrow H_n(C_\bullet) = 0. \end{cases}$

Definition: Let $C_\bullet$ and $D_\bullet$ be two complexes of $R$-modules

$u_\bullet : C_\bullet \to D_\bullet$ is a morphism of complexes:

$$u = \{u_n : C_n \to D_n\}_{n\in\mathbb{Z}} \text{ st}$$
$$u_n \circ d_{n+1}^C = d_{n+1}^D \circ u_{n+1} \quad \forall n\in\mathbb{Z} \quad (ud = du)$$

$$\begin{array}{ccccccccccc}
C_\bullet \cdots & \to & C_{n+2} & \to & C_{n+1} & \xrightarrow{d_{n+1}^c} & C_n & \xrightarrow{d_n^c} & C_{n-1} & \to & \cdots \\
\downarrow u_\bullet & & \downarrow & /\!/\!/ & \downarrow u_{n+1} & /\!/\!/ & \downarrow u_n & /\!/\!/ & \downarrow u_{n-1} & & \\
D_\bullet \cdots & \to & D_{n+2} & \to & D_{n+1} & \xrightarrow[d_{n+1}^D]{} & D_n & \xrightarrow[d_n^D]{} & D_{n-1} & \to & \cdots
\end{array}$$

$u$ is called isomorphism $\Leftrightarrow$ $u_n$ isomorphism $\forall n\in\mathbb{Z}$

$u : C_\bullet \to D_\bullet$ induces $H_\bullet(u_\bullet) : H_\bullet(C_\bullet) \to H_\bullet(D_\bullet)$

as follows: $\quad H_n(u) : H_n(C_\bullet) \to H_n(D_\bullet)$

$$\operatorname{cls}(c) \longmapsto \operatorname{cls}(u_n(c))$$

$$c \in Z_n(C) \subseteq C_n$$

Exercise: $H_\bullet(u_\bullet)$ is a well defined morphism of complexes (graded modules).

<u>Definition</u> : $u_\bullet$ is called quasi-isomorphism (quism) if $H_\bullet(u_\bullet)$ is isom.

<u>Remark</u> : $0 \longrightarrow C_\bullet$ quism $\Leftrightarrow$ $C_\bullet$ is acyclic

<u>Example</u> :

$$\begin{tikzcd}
C_\bullet = \cdots \arrow[r] \arrow[d, equal] & \mathbb{Z}_4 \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z}_4 \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z}_4 \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z}_4 \arrow[r] \arrow[d, two heads] & 0 & \hat{x} \arrow[d] \\
D_\bullet = \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0 & \bar{x}
\end{tikzcd}$$

is a quism.

<u>Definition</u> :

<u>Cochain complex</u> : $C^\bullet$ $\qquad C^n = C_{-n}$

$$d_C^n : C^n \longrightarrow C^{n+1}, \quad d^2 = 0$$

<u>Definition</u> : The chain complex $C_\bullet$ is called :

<u>bounded</u> if $C_n = 0 \ \forall\ |n| >> 0$ $\quad (\operatorname{ampl}(C_\bullet) < \infty) \in \mathbf{Ch}_b$

<u>bounded above</u> $C_n = 0 \ \forall\ n >> 0$ $\quad (\sup(C_\bullet) < \infty) \in \mathbf{Ch}_-$

<u>bounded below</u> $C_n = 0 \ \forall\ n << 0$ $\quad (\inf(C_\bullet) > -\infty) \in \mathbf{Ch}_+$

$$\sup(C_\bullet) = \sup\{n \mid C_n \neq 0\}$$
$$\inf(C_\bullet) = \inf\{n \mid C_n \neq 0\}$$
$$\operatorname{ampl}(C_\bullet) = \sup(C_\bullet) - \inf(C_\bullet)$$

$$\mathbf{Ch}_{\geq 0} = \{C_\bullet \mid C_n = 0 \ \forall\, n < 0\}$$

<u>Optional</u> : <u>Definition</u> The cochain complex $C_\bullet$ is called :

<u>bounded above</u> if $C^n = 0 \ \forall\ n >> 0$ ($\Leftrightarrow C_\bullet$ bounded below)

<u>bounded below</u> if $C^n = 0 \ \forall\ n << 0$ ($\Leftrightarrow C_\bullet$ —— above).

## ② Abelian categories

Recall: $\mathcal{C}$ is called category if $Obj(\mathcal{C})$ class

$Mor(\mathcal{C}) = \{Hom(A,B) \mid A,B \in \mathcal{C}\}$ sets

$\in \mathcal{C}$ set

$f : A \to B$

s.t.

① $\forall A \quad \exists \; id_A \in Hom_{\mathcal{C}}(A,A)$, called identity.

② $\exists \; Hom_{\mathcal{C}}(A,B) \times Hom_{\mathcal{C}}(B,C) \longrightarrow Hom_{\mathcal{C}}(A,C)$ composition map.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

- With the axioms:

associativity axiom: $\forall \; A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, $h(gf) = (hg)f$ $hgf$

unit axiom: $\forall \quad A \xrightarrow{id_A} A \xrightarrow{f} B \xrightarrow{id_B} B$, $id_B \circ f = f = f \circ id_A$.

Definition $\mathcal{C}$ is called small if $Obj(\mathcal{C})$ is a set

Examples

Definition $f : A \to B$ is called isomorphism if $\exists g : B \to A$ st $fg = id_B$, $gf = id_A$ ($\exists!$)

① Sets

② $Ab$ = category of abelian groups

③ R-mod, mod-R

④ Discrete category: $X$ set (or class) (small) $\to X$ set.

$Obs(X) = X$

$Mor(X)$: $Hom_X(a,b) = \begin{cases} id_a, & a = b \\ \emptyset, & a \neq b \end{cases}$

⑤ $\mathcal{C} = (\{*\}, Mor\{*\} = M)$

M monoid (small)

$(M, \cdot)$, id, assoc.

⑥ $\mathcal{P}$ poset category: $Obj(\mathcal{P}) = P$, $\leq$ reflexive, transitive, antisymmetric

$Mor(\mathcal{P})$: $Hom_{\mathcal{P}}(p,q) = \begin{cases} p \leq q, & p \leq q \\ \emptyset, & p \not\leq q \end{cases}$

(small)

Definition:

$\mathcal{A}$ category $\mathcal{A}$ is called Ab-category if

(1) $\mathrm{Hom}(A,B)$ is an abelian group $(+)$

(2) $\forall^{\mathcal{A}}$ $A \xrightarrow{f} B \underset{g'}{\overset{g}{\rightrightarrows}} C \xrightarrow{h} D$

$$h(g+g')f = hgf + hg'f$$

Remark: $\mathrm{Hom}_{\mathcal{A}}(A,A)$ is an associative ring $\forall A \in \mathcal{A}$. $\Rightarrow$ $A$-$\{*\}$ $\Leftrightarrow$ ring.

Example: Ab is an Ab-category, mod-R, ...

Lemma: Ch is an Ab-category. $\boxed{\text{mod-R} \rightsquigarrow Ch(\text{mod-R})}$

Proof: $Obj(Ch) = \{ C_\bullet \mid$ chain complexes of R-modules $\}$

$Mor(Ch) = \{ \mathrm{Hom}_{Ch}(C_\bullet, D_\bullet) = \{u: C_\bullet \to D_\bullet \mid u_\bullet$ morphism$\}$.

$$\left.\begin{aligned} f &: C_\bullet \to D_\bullet \\ g &: C_\bullet \to D_\bullet \end{aligned}\right\} \Rightarrow f+g : C_\bullet \to D_\bullet$$

$$(f+g)_n(x) = f_n(x) + g_n(x)$$

$0 : C_\bullet \to D_\bullet$

$-f : C_\bullet \to D_\bullet$

(1) and (2) are satisfied! ∎

Let $\mathcal{C}$ be a category.

- $I \in Obs(\mathcal{C})$ is called intial object if $\mathrm{Hom}_{\mathcal{C}}(I,C)$ has only one element $\forall C \in \mathcal{C}$.
- $T \in Obs(\mathcal{C})$ is called terminal object if $\mathrm{Hom}_{\mathcal{C}}(C,T)$ —"—
- zero object $0 \in \mathcal{C}$ = terminal + initial obj.

Examples: $\nexists\, 0$ in Sets

$\exists\, 0$ in Ab, R-mod, mod-R, ...

Definition: $A \times B$ is called direct product and satisfies:

$$\begin{tikzcd} C \arrow[r, dotted, "\alpha"] \arrow[rr, bend left, "\alpha_2"] \arrow[dr, "\alpha_1"'] & A \times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \\ & A & \end{tikzcd}$$

direct sum:

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[rr, bend left, "\forall \alpha_1"] & A \oplus B \arrow[r, dotted, "\exists! \alpha"] & C \\ & B \arrow[u, "i_2"] \arrow[ur, "\forall \alpha_2"'] & \end{tikzcd}$$

Let $\mathcal{C}$ with $0$. Rmk: $\mathrm{Hom}_{\mathcal{C}}(B,C)$ $\exists$ $B \to 0 \to C$

$$\begin{tikzcd} B \arrow[r] \arrow[rr, bend right, "0"'] & 0 \arrow[r] & C \end{tikzcd}$$

Zero map.

Definition: $f : B \to C$

$i : A \to B$ is called Kernel of $f$ if it satisfies

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[rr, bend left, "0 = fi"] & B \arrow[r, "f"] & C \\ \forall A' \arrow[u, dashed, "\exists! e'"] \arrow[ur, "e"'] & & \end{tikzcd} \quad e = ie', \quad fe = 0$$

{A subobject of B denoted $\ker(f)$}

$\ker(f)$ is unique up to isomorphism.

$p : C \to D$ is called Cokernel of $f$ if it satisfies

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[rr, bend left, "pf = 0"] & C \arrow[r, "p"] \arrow[dr, "\forall g"'] & D \arrow[d, dashed, "\exists! g'"] \\ & & D' \end{tikzcd} \quad gf = 0$$

$\mathrm{Coker}(f)$ is unique up to isomorphism,

Let $\mathcal{C}$ be any category

$A \underset{e_2}{\overset{e_1}{\rightrightarrows}} B \xrightarrow{f} C$ is called monic if $\forall e_1, e_2$ $fe_1 = fe_2 \Rightarrow e_1 = e_2$.

(If $\mathcal{A}$ is an Ab-category $f$ monic $\Leftrightarrow fe = 0 \Rightarrow e = 0$)

$B \xrightarrow{f} C \underset{p_2}{\overset{p_1}{\rightrightarrows}} D$ is called epi if $\forall p_1, p_2$ $p_1 f = p_2 f \Rightarrow p_1 = p_2$

(If $\mathcal{A}$ is an Ab-category $f$ epi $\Leftrightarrow pf = 0 \Rightarrow p = 0$).

END of Lesson #1.

Exercise: Let $\mathcal{C}$ be a category with $0$ object. then for $\forall f : B \to C$

(1) $\ker(f)$ is monic.

(2) $\mathrm{Coker}(f)$ is epi.

Definition: A category $\mathcal{A}$ is called additive if

(1) $\mathcal{A}$ is an Ab category (2) $0 \in \mathcal{A}$ (3) $\exists A \times B$

Exercise $A \times B \cong A \oplus B$

Definition: A category $\mathcal{A}$ is called abelian if

(1) $\mathcal{A}$ is an additive category

(2) $\forall f \in \mathrm{Hom}(B,C)$ $\exists$ $\ker(f)$, $\mathrm{Coker}(f)$

(3) $\overset{\mathcal{A}}{\forall}$ $f$ monic $= \ker(\mathrm{coker} f)$

$f$ $\forall$ epi $= \mathrm{Coker}(\ker f)$

Exercise: If $\mathcal{A}$ is an abelian category, ker = monic, coker = epi.

Example: mod-$R$, $R$-mod.

Theorem If $\mathcal{A}$ is an abelian category, then $Ch(\mathcal{A})$ is an abelian category.

Exercise $f$ isomorphism $\Leftrightarrow$ $f$ is monic and epi