Homework problems, lessons #1-22

Homological Algebra, Spring 2006

Lesson #1

(1) In a category $\mathcal{C}$, with 0 object every cokernel is epi.
(2) Give an example of a category where there exists an epi that is not a cokernel.

Lesson #2

(1) In an additive category $\mathcal{A}$, for every $A, B \in Obj(\mathcal{A})$ we have $A \oplus B \cong A \times B$.
(2) Show that in an **Ab**-category, for a morphism $f\colon B \to C$, the following hold:
  (a) f is monic if and only if $\operatorname{Ker} f = 0$.
  (b) f is epi if and only if $\operatorname{Coker} f = 0$.
  Do we need a 0 object in $\mathcal{A}$? Explain.
(3) Let $\mathcal{A}$ be an abelian category. A morphism $f\colon B \to C$ is an isomorphism if and only if it is monic and epi.
(4) Find an example of an additive category that is not abelian. Justify your answer.
(5) Without using that $\mathbf{Ch}(\mathcal{A})$ is an abelian category, show that a morphism $f_\bullet\colon B_\bullet \to C_\bullet$ is epi if and only if $\operatorname{Coker}(\operatorname{Ker} f_\bullet) = f_\bullet$.
(6) Exercise A 1.1 page 419.[1]
(7) Exercise A.1.2 page 419.

Lesson #3

(1) Let $\mathcal{C}$ be any category and let $\mathcal{A}$ be an abelian category. Show that $\mathcal{A}^{\mathcal{C}}$ is an abelian category.
(2) Lemma 1.6.2 page 25.
(3) Exercise 1.6.1, page 26.
(4) Exercise 1.6.2, page 26.
(5) Exercise 1.6.3, page 27.
(6) Example A.4.4, page 426.
(7) Example 1.6.3.(1), page 25.

Lesson #4

(1) Exercise 1.3.3, page 13.
(2) Exercise 1.3.4, page 15.
(3) Exercise 1.3.5, page 15.
(4) Let $f_\bullet\colon B_\bullet \to C_\bullet$ be a morphism of complexes. There exists a short exact sequence of complexes

$$0 \to C_\bullet \to \operatorname{Cone}(f_\bullet) \to B_\bullet[-1] \to 0.$$

Lesson #5

(1) Let $\mathcal{A}$ be an abelian category and let $C_\bullet$ be a complex in $\mathbf{Ch}(\mathcal{A})$. Show that the following are equivalent:
  (i) $C_\bullet$ is split and exact.
  (ii) $C_\bullet \cong \operatorname{Cone}(\iota)$ and $\iota\colon B_\bullet \to Z_\bullet$ is an isomorphism.
  (ii′) $C_\bullet \cong \operatorname{Cone}(\mathrm{id}^{B_\bullet})$
  (iii) There exists maps $s_n\colon C_n \to C_{n-1}$ such that $sd + ds = \mathrm{id}^{C_\bullet}$.
(2) Exercise 1.5.2, page 19.
(3) Exercise 1.5.3, page 21.

[1]The exercises are from: *An introduction to the homological algebra* by C.A. Weibel, Cambridge Studies. in Advanced Math., vol. 38, Cambridge Univ. Press, 1995.

(4) Exercise 1.5.4, page 21.

Some of the following problems will be needed in the Derived Category section:

(5) Exercise 1.5.5, page 23.
(6) Exercise 1.5.6, page 24.
(7) Exercise 1.5.7, page 24.
(8) Exercise 1.5.8, page 24.
(9) Exercise 1.5.9, page 24.
(10) Exercise 1.5.10, page 24.

Lesson #6

(1) Let $\mathcal{A}$ be an abelian category. Is $\mathbf{K}(\mathcal{A})$ an abelian category ? Justify your answer.
(2) Let $\mathcal{A}$ be an abelian category and let $g_\bullet, g'_\bullet \colon C_\bullet \to D_\bullet$, $f_\bullet \colon B_\bullet \to C_\bullet$ and $h_\bullet \colon D_\bullet \to E_\bullet$ be morphisms in $\mathbf{Ch}(\mathcal{A})$. Show that if $g_\bullet \sim g'_\bullet$, then $h_\bullet g_\bullet f_\bullet \sim h_\bullet g'_\bullet f_\bullet$.
(3) Without using that $\mathbf{K}(\mathcal{A})$ is an additive category, show that the direct product $C_\bullet \times D_\bullet$ of any two objects in $\mathbf{K}(\mathcal{A})$ exists.
(4) Let $\mathcal{A}$ be an abelian category and let $\{P_\alpha\}_{\alpha\in\Lambda}$ be a family of projective objects such that $P = \coprod_{\alpha\in\Lambda} P_\alpha$ exists. Show that P is also a projective object.
(5) Exercise 2.2.1, page 34.

Lesson #7

(1) Exercise 2.2.1 page 34.
(2) Exercise 2.2.2 page 34.
(3) Exercises 2.3.2 page 39 and 2.3.5 page 42.
(4) Exercise 2.3.4 page 40.
(5) Exercise 2.3.6 page 42.
(6) Exercise 2.3.7 page 43.
(7) Exercise 2.3.8 page 43.

Lesson #8

(1) Exercise 2.2.4 page 38.
(2) Exercise 2.1.1 page 31.
(3) Exercises 2.1.2 page 32.
(4) Exercise 2.4.2 page 45.
(5) Exercise 2.4.3 page 47.

Lesson #9

(1) Exercise 2.4.4 page 49.
(2) Exercise 2.5.1 page 50.
(3) Exercise 2.5.1 page 50.
(4) Exercise 2.5.2 page 50.
(5) Exercise 2.5.3 page 51dvi.

Lesson #10

(1) Let $\alpha\colon C_\bullet \to C'_\bullet$ and $\beta\colon D_\bullet \to D'_\bullet$ be morphisms of complexes of right and respectively left R-modules. Then the following hold:
   (a) $C_\bullet \otimes_R \mathrm{Cone}(\beta) \cong \mathrm{Cone}(C_\bullet \otimes_R \beta)$.
   (b) $\mathrm{Cone}(\alpha) \otimes_R D_\bullet \cong \mathrm{Cone}(\alpha \otimes_R D_\bullet)$.
(2) Let $\alpha\colon C_\bullet \to C'_\bullet$ and $\beta\colon D_\bullet \to D'_\bullet$ be morphisms of complexes of right R-modules. Then the following hold:

(a) $\mathrm{Hom}_R(C_\bullet, \mathrm{Cone}(\beta)) \cong \mathrm{Cone}(\mathrm{Hom}_R(C_\bullet, \beta))$.
(b) $\mathrm{Hom}_R(\mathrm{Cone}(\alpha), D_\bullet) \cong \mathrm{Cone}(\mathrm{Hom}_R(\alpha[-1], D_\bullet))$.

(3) Exercise 1.2.5 page 9.
(4) Exercise 1.2.6 page 9.
(5) Exercise 1.2.8 page 10.
(6) Exercise 2.7.1 page 61.
(7) Exercise 2.7.2 page 62.
(8) Exercise 2.7.3 page 63.
(9) Exercise 2.7.4 page 64.
(10) Exercise 2.7.5 page 65.

Lesson #11

(1) Exercise 2.6.1 page 53.
(2) Exercise 2.6.2 page 54.
(3) Exercise 2.6.3 page 54.
(4) Exercise 2.6.4 page 54.
(5) Exercise 2.6.5 page 57.
(6) Exercise 2.6.6 page 58.

Lesson #12

(1) Exercise 3.1.1 page 68.
(2) Exercise 3.1.2 page 68.
(3) Exercise 3.1.3 page 68.
(4) Exercise 3.1.4 page 68.
(5) Exercise 3.2.1 page 69.
(6) Exercise 3.2.2 page 69.
(7) Exercise 3.2.5 page 70.

Lesson #13

(1) (Künneth formula for Hom) Let $P_\bullet$ and $Q_\bullet$ be complexes of right R-modules. Assume that $P_\bullet$ is a complex of projective modules. Then, the following hold.
(a) If $\mathrm{d}^{P_\bullet}{=}0$, then for each $n \in \mathbb{Z}$, there exists a natural isomorphism of abelian groups:

$$\alpha\colon \mathrm{H}_n\left(\mathrm{Hom}_R(P_\bullet, Q_\bullet)\right) \xrightarrow{\cong} \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(P_i, \mathrm{H}_{i+n}\left(Q_\bullet\right))$$

(b) If $B_\bullet(P_\bullet)$ is a complex of projectives, then for each $n \in \mathbb{Z}$ there exists an exact sequence of abelian groups

$$0 \to \prod_{i\in\mathbb{Z}} \mathrm{Ext}^1_R(\mathrm{H}_i\left(P_\bullet\right), \mathrm{H}_{i+n+1}\left(Q_\bullet\right)) \to \mathrm{H}_n\left(\mathrm{Hom}_R(P_\bullet, Q_\bullet)\right) \to \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(P_i, \mathrm{H}_{i+n}\left(Q_\bullet\right)) \to 0$$

(2) Example 3.3.3 page 73.
(3) Exercise 3.3.1 page 74[2].
(4) Exercise 3.3.2 page 74.
(5) Let R be a ring and let A, B, $\{A_i\}_{i\in I}$, $\{B_i\}_{i\in I}$ be right R-modules. Then the following hold.

(a) $\mathrm{Hom}_R(\bigoplus_{i\in I} A_i, B) = \prod_{i\in I} \mathrm{Hom}_R(A_i, B)$.

(b) $\mathrm{Hom}_R(A, \prod_{i\in I} B_i) = \prod_{i\in I} \mathrm{Hom}_R(A, B_i)$.

[2]Consult the file with book corrections.

(6) Let $\{\boldsymbol{C}_i\}_{i\in I}$ be a family of cochain complexes of R-modules then for all $n \in \mathbb{Z}$ there is an isomorphism of R-modules

$$\mathrm{H}^n\,(\prod_{i\in I}\boldsymbol{C}_i) \cong \prod_{i\in I}\mathrm{H}^n\,(\boldsymbol{C}_i)$$

Lesson #14

(1) Exercise 3.5.1 page 82.
(2) Exercise 3.5.2 page 83.
(3) Example 3.5.3 page 82 and Exercise 3.5.3 page 85.
(4) Exercise 3.5.4 page 86.
(5) Exercise 3.5.4 page 86.

Lesson #15

(1) Let A be a right R-module such that $A = \bigcup_{i\geq 0} A_i$, where

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_i \subseteq A_{i+1} \subseteq \cdots \subseteq A$$

are right R-submodules of A and let B be another right R-module. Then there is an equality:

$$\varprojlim \mathrm{Hom}_R(A_i, B) = \mathrm{Hom}_R(A, B).$$

(2) Exercise 4.1.1 page 93.
(3) Exercise 4.1.2 page 95.
(4) Example 4.1.3 page 95.

Lesson #16

(1) Exercise 4.3.1, page 100.
(2) Exercise 4.3.2, page 101.
(3) Example 4.3.3 -*First Theorem*, page 104.
(4) Example 4.3.3 -*Second Theorem*, page 104.

Lesson #18

(1) Let $F_\bullet$ be a bounded below complex of finite free modules over a local ring $(R, \mathfrak{m}, k)$. Then, the following conditions are equivalent.
   (i) $F_\bullet$ is minimal (i.e. $d^{F_\bullet}(F) \subseteq F$).
   (ii) Each quasi-isomorphism $\alpha_\bullet : F_\bullet \to F_\bullet$ is an isomorphism.
   (iii) Each quasi-isomorphism $\beta_\bullet \colon F_\bullet \to F'_\bullet$ to a bounded below minimal complex of free modules is an isomorphism.
   (iv) Each quasi-isomorphism $\gamma\colon F_\bullet \to G_\bullet$ to a bounded below complex of free modules is injective, and $G_\bullet = \mathrm{Im}\,\gamma \oplus E_\bullet$ for a split-exact subcomplex $E_\bullet$.
(2) Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring and let $A \neq 0$ be a finitely generated R-module. Show that if $\mathrm{depth}(A) = 0$, then there exists $a \in A$ a non zero element such that $\mathfrak{m}a = 0$.
(3) Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring and let A be a finitely generated R-module. If $\boldsymbol{x}$ is an A-regular sequence of length n, then

$$\mathrm{pd}_R\,(A/\boldsymbol{x}A) = \mathrm{pd}_R\,(A) + n.$$

(4) Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring. If N is the nth syzygy of a finitely generated R- module in a finite free resolution, then

$$\mathrm{depth}(N) \geq \min\{n, \mathrm{depth}(R)\}.$$

(5) Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a homomorphism of local rings such that $\varphi(\mathfrak{m}) \subseteq \varphi(\mathfrak{n})$. If A is a finitely generated S-module, then

$$\operatorname{depth}_R(A) = \operatorname{depth}_S(A).$$

Lesson #19

(1) Let $(R,\mathfrak{m},k)$ be a local commutative noetherian ring such that $\operatorname{depth}(R) \geq 1$. Show that there exists an R-regular element x such that $x \in \mathfrak{m} \setminus \mathfrak{m}^2$.
(2) Exercise 4.4.1 page 106.
(3) Exercise 4.4.2 page 106.

Lesson #20

(1) Exercise 4.5.1, page 112.
(2) Exercise 4.5.2, page 113.
(3) Exercise 4.5.3, page 113.
(4) Exercise 4.5.4, page 113.
(5) Let $(R,\mathfrak{m},k)$ be a commutative, noetherian, local ring and let $A \neq 0$ be a finite R-module. For a sequence $\boldsymbol{x} = x_1, \ldots x_n \subset \mathfrak{m}$ the following are equivalent:
   (a) $\mathrm{H}_i\,(\boldsymbol{x}; A) = 0$ for all $i \geq 1$.
   (b) $\mathrm{H}_1\,(\boldsymbol{x}; A) = 0$.
   (c) $\boldsymbol{x}$ is a maximal A-regular sequence.
   (d) $\boldsymbol{x}$ is an A-regular sequence.

Lesson #21

(1) Exercise 5.1.2(1), page 121.
(2) Exercise 5.1.2(2 and 3), page 121.
(3) Exercise 5.1.3, page 121.
(4) Exercise 5.2.2, page 124.

Lesson #22

(1) Exercise 5.4.1, page 134.
(2) Exercise 5.4.2, page 134.
(3) Exercise 5.4.3, page 135.
(4) Exercise 5.4.4, page 135.