CONSTRUCTION OF MODULES WITH FINITE HOMOLOGICAL DIMENSIONS

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ABSTRACT. A new homological dimension, called G*-dimension, is defined for every finitely generated module M over a local noetherian ring R. It is modeled on the CI-dimension of Avramov, Gasharov and Peeva and has parallel properties. In particular, a ring R is Gorenstein if and only if every finitely generated R-module has finite G*-dimension. The G*-dimension lies between the CI-dimension and the G-dimension of Auslander and Briger. This relation belongs to a longer sequence of inequalities, where a strict inequality in any place implies equalities to its right and left. Over general local rings, we construct classes of modules that show that a strict inequality can occur at almost every place in the sequence.

INTRODUCTION

We study various homological dimensions, defined for finitely generated modules M over a commutative noetherian local ring R, and refining the classical projective dimension, $\operatorname{pd}_R M$.

One theme is given by a famous result of Auslander, Buchsbaum and Serre: the ring R is regular if and only if every finitely generated R-module has finite projective dimension. Auslander and Briger [1] introduced a notion of Gorenstein dimension, denoted G-dim. It has the same relation to the Gorenstein property of R as the projective dimension has to its regularity. A dimension for modules that corresponds to the complete intersection property of rings was defined by Avramov, Gasharov and Peeva [8]. It is called complete intersection dimension and is denoted CI-dim. Another dimension which reflects the complete intersection property of the ring was defined by Gerko [14]. He calls it polynomial complete intersection dimension and denotes PCI-dim; for uniformity of notation, in this paper we call it lower complete interection dimension and denote CI_*-dim. Also [14] Gerko introduced a dimension, called Cohen-Macaulay dimension and denoted CM-dim, that reflects the Cohen-Maculay property of rings.

A first goal of the paper is to introduce and study a new dimension, called *upper Gorenstein dimension* or G*-*dimension*, and denoted G*-dim. It is modeled on CI-dimension and has the expected basic properties parallel to those of CI-dimension. In particular, the ring R is Gorenstein if and only if every finitely generated R-module M has finite G*-dimension. For each finite module M over a noetherian ring R, $\text{G}^*\text{-dim}_R M$ fits into the following scheme of inequalities

$$\text{CM-dim}_R M \leq \text{G-dim}_R M \leq \left\{ \begin{matrix} \text{CI}_*\text{-dim}_R M \\ \text{G}^*\text{-dim}_R M \end{matrix} \right\} \leq \text{CI-dim}_R M \leq \operatorname{pd}_R M.$$

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If one of these dimensions is finite, then it is equal to those to its left. The statements not involving G*-dimension are proved in [1], [8] and [14].

A second goal of the paper is to demonstrate that the homological dimensions described above can be used effectively to distinguish between modules over general local rings. More precisely, we prove the following result.

Main Theorem *Let $(Q,\boldsymbol{n})$ be a local ring, J an ideal of Q contained in $\boldsymbol{n}^2$, and $R = Q/J$. The following hold:*

(1) *If J is perfect, not complete intersection, and $\operatorname{grade} J = 2$, then there exist R-modules M with*
$$0 = \text{CM-dim}_R M < \text{G-dim}_R M = \infty.$$
(2) *If J is Gorenstein, not complete intersection, and $\operatorname{grade} J = 3$, then there exist R-modules M with*
$$0 = \text{G}^*\text{-dim}_R M < \text{CI}_*\text{-dim}_R M = \infty.$$
(3) *If J is complete intersection and $\operatorname{grade} J \geq 1$, then there exist R-modules M with*
$$0 = \text{CI-dim}_R M < \operatorname{pd}_R M = \infty.$$

If, furthermore Q is an algebra over a field l, then the following also hold.

(4) *If $\operatorname{depth} Q \geq 4$, then there exists a perfect ideal J with $\operatorname{grade} J = 4$ and an R-module M with*
$$0 = \text{CI}_*\text{-dim}_R M < \text{CI-dim}_R M = \infty.$$
(5) *If $\operatorname{card}(l) > 3$ and $\operatorname{depth} Q \geq 5$, then there exists a Gorenstein ideal J with $\operatorname{grade} J = 5$ and an R-module M with*
$$0 = \text{G}^*\text{-dim}_R M = \text{CI}_*\text{-dim}_R M < \text{CI-dim}_R M = \infty.$$

The proof is given in Sections 4 and 5. Part (3) is already known, see [8, (3.1)], but the proof here is different. Part (4) answers a question of Gerko [14, (2.9)]. We do not know whether there exist a ring R and an R-module M with $\text{G-dim}_R M < \text{G}^*\text{-dim}_R M$ or with $\text{CI}_*\text{-dim}_R M < \text{G}^*\text{-dim}_R M$. A positive answer to the second question would give a positive answer to the first one; a good candidate for this could be a ring and a module constructed for Part (4).

The idea used in proving the Main Theorem is the following. From a suitable polynomial ring defined over $\mathbb{Z}$ or over a field l we construct, by factoring out a suitable ideal, a ring B characterised by the finiteness of the first homological dimension. We then choose a B-module for which the second dimension is finite. Techniques from the Section 3 allow us to construct from this B-module, an R-module preserving both homological dimensions.

In Section 1 we recall definitions and results needed in the rest of the paper. In Section 2 we introduce and study G*-dimension. In Section 3 we study the behavior of homological dimensions under certain types of change of rings.

1. Homological dimensions

All rings considered are commutative and noetherian. In this section $(R,\boldsymbol{m},k)$ denotes a local ring R with maximal ideal $\boldsymbol{m}$ and residue field $k = R/\boldsymbol{m}$. The *embedding dimension* of R, denoted $\operatorname{edim} R$, is the minimal number of generators of the maximal ideal $\boldsymbol{m}$. We set $\operatorname{codepth} R = \operatorname{edim} R - \operatorname{depth} R$.

Let M denote a finitely generated R-module. The n'th *Betti number* of M over R is defined by $\beta_n^R(M) = \operatorname{rank}_k(\operatorname{Ext}_R^n(M,k))$. The *Poincaré series* of M over R is the formal power series

$$P_M^R(t) = \sum_{n=0}^{\infty} \beta_n^R(M)t^n.$$

The *complexity* of M is defined by Avramov [3] as

$$\operatorname{cx}_R M = \inf \left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{c} \text{there exists a non-zero constant } a \\ \text{such that } \beta_n^R(M) \leq a n^{d-1} \text{ for } n \gg 0 \end{array} \right\}.$$

We recall the definitions of various homological dimensions for R-modules. They are of two types. A first one is defined in terms of lengths of resolution by modules of a certain class whose members are declared to have homological dimension 0. A second type is defined in terms of homological dimensions of the first kind, over an appropriate deformations of some flat extension of R. We set $(\ )^* = \operatorname{Hom}_R(\ ,R)$.

The module M has *G-dimension* 0 if the following conditions are satisfied:

(i) $M \cong M^{**}$.
(ii) $\operatorname{Ext}_R^i(M,R) = 0$, for all $i > 0$.
(iii) $\operatorname{Ext}_R^i(M^*,R) = 0$, for all $i > 0$.

The *Gorenstein dimension* of M, defined by Auslander and Bridger [1] and denoted $\operatorname{G-dim}_R M$, is the least number n for which there exists an exact sequence

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

where G_i has G-dimension 0 for $i = 0, \ldots, n$.

In [14] Gerko introduces a *lower complete intersection dimension*, denoted $\operatorname{CI}_*\text{-}\dim_R M$, as the least number n for which there exists an exact sequence

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

where $\operatorname{G-dim}_R G_i = 0$ and $\operatorname{cx}_R G_i < \infty$ for $i = 0, \ldots, n$.

The following property follows by induction on $\operatorname{CI}_*\text{-}\dim_R M$.

1.1. $\operatorname{CI}_*\text{-}\dim_R M < \infty$ if and only if $\operatorname{G}^*\text{-}\dim_R M < \infty$ and $\operatorname{cx}_R M < \infty$.

The *grade* of an R-module M is defined by the formula

$$\operatorname{grade} M = \inf\{i \mid \operatorname{Ext}_R^i(M,R) \neq 0\}.$$

When I is an ideal of R, abusing notation we write $\operatorname{grade}_R I$ or $\operatorname{grade} I$ to denote the grade of the R-module R/I. The following equality is proved in [1, p. 95]:

1.2. If $\operatorname{G-dim}_R M < \infty$ then $\operatorname{G-dim}_R M = \sup\{n \mid \operatorname{Ext}_R^i(M,R) \neq 0\}$.

From the definition of grade and 1.2 we see that:

1.3. If $M \neq 0$, then there is an inequality $\operatorname{grade}_R M \leq \operatorname{G-dim}_R M$.

An R-module module M is called *perfect* (respectively, *G-perfect*) if $\operatorname{grade}_R M = \operatorname{pd}_R M$ (respectively, $\operatorname{grade}_R M = \operatorname{G-dim}_R M$).

Let Q be a local ring and J an ideal of Q. Abusing terminology, we say that J is *perfect* (respectively, *G-perfect*) if the Q-module Q/J has the corresponding property. The ideal J is called *Gorenstein* if it is perfect and $\beta_g^Q(Q/J) = 1$ for $g = \operatorname{grade}_Q J$. The following property of grade is from [6, (2.7)].

1.4. Let Q be a ring, let J be an ideal of finite projective dimension, and set $R = Q/J$. For every R-module M there are inequalities

$$\operatorname{grade}_R M + \operatorname{grade}_Q R \leq \operatorname{grade}_Q M \leq \operatorname{grade}_R M + \operatorname{pd}_Q R$$

which become equalities when J is perfect.

We say that R has a *CI-deformation* (respectively, *G-deformation*) if there exist a local ring Q and a complete intersection (respectively, G-perfect) ideal J in Q such that $R = Q/J$. A *CI-quasi-deformation* (respectively, *G-quasi-deformation*) of R is a diagram of local homomorphisms $R \to R' \leftarrow Q$ with $R \to R'$ a flat extension and $Q \to R'$ a CI-deformation (respectively, G-deformation). We set $M' = M \otimes_R R'$.

The *complete intersection dimension* of M is defined by Avramov, Gasharov and Peeva [8] by the formula

$$\text{CI-dim}_R M = \inf \left\{ \operatorname{pd}_Q M' - \operatorname{pd}_Q R' \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \quad \text{is a} \\ \text{CI-quasi-deformation} \end{array} \right\}.$$

The *Cohen-Macaulay dimension* of M is introduced by Gerko [14] by the formula

$$\text{CM-dim}_R M = \inf \left\{ \text{G-dim}_Q M' - \text{G-dim}_Q R' \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \quad \text{is a} \\ \text{G-quasi-deformation} \end{array} \right\}.$$

For the sake of uniformity of notation, we sometimes write $\text{P-dim}_R M$ for the classical projective dimension. We then have notions of *homological dimension* of M, denoted $\text{H-dim}_R M$, for H=P, CI, CI_*, G or CM. We say that R has property (H) with H=P (respectively, CI, CI_*, G or CM), when R is regular (respectively, complete intersection, complete intersection, Gorenstein, Cohen-Macaulay).

The next four assertions are classical for H=P.

1.5. If $\text{H-dim}_R M < \infty$, then $\text{H-dim}_R M + \operatorname{depth}_R M = \operatorname{depth} R$.

For proofs see [8, (1.4)] (respectively, [14, (2.7)], [1, (4.13.b)] , [14, (3.8)]) when H=CI (respectively, CI_*, G, CM).

1.6. For each $n \geq 0$ there is an equality

$$\text{H-dim}_R \operatorname{Syz}_n^R(M) = \max\{\text{H-dim}_R M - n,\ 0\}.$$

A proof for H=CI is given in [8, (1.9.1)]; similar arguments work for H= CI_*, G, CM.

1.7. The following conditions are equivalent:

(i) The ring R has property (H).
(ii) $\text{H-dim}_R M < \infty$ for every R-module M.
(iii) $\text{H-dim}_R M = 0$ for every R-module M with $\operatorname{depth}_R M \geq \operatorname{depth} R$.
(iv) $\text{H-dim}_R k = \operatorname{depth} R$.
(v) $\text{H-dim}_R k < \infty$.

For proofs of (i) $\iff$ (ii) $\iff$ (v) see [8, (1.3)] (respectively, [14, (2.5)], [1, (4.20)], [14, (3.9)]) when H=CI (respectively, CI_*, G, CM). For (ii) $\implies$ (iii) use 1.5 and for (iii) $\implies$ (iv) use the Depth Lemma [4, (1.2.6)] and 1.6.

1.8. If $R \to R'$ is a local flat extension and $M' = M \otimes_R R'$, then

$$\text{H-dim}_R M \leq \text{H-dim}_{R'} M'$$

with equality when H=P, CI_* or G.

For a proof when H=CI see [8, (1.13)(1)]; a similar proof works for H=CM. For a proof when H=G see [5, (4.1.4)]; using, in addition, [4, (4.2.5)(2)] and Proposition 1.9 below, we get the equality for H=CI_*.

It follows from [14, (3.2)], by definition, from [14, (2.6)] and respectivelly from [8, (1.4)] that the different homological dimensions above are related as follows:

1.9. There are inequalities

$$\text{CM-dim}_R M \le \text{G-dim}_R M \le \text{CI}_*\text{-dim}_R M \le \text{CI-dim}_R M \le \mathrm{pd}_R M$$

and if one of these dimensions is finite, then it is equal to those to its left.

We recall one further property of G-dimension, which is a consequence of [15, Prop. 5] (cf. also [5, (7.11)]).

1.10. If $R = Q/J$ where Q is a local ring and J is a Gorenstein ideal, then $\text{G-dim}_Q M < \infty$ if and only if $\text{G-dim}_R M < \infty$; when this holds,

$$\text{G-dim}_R M + \mathrm{pd}_Q R = \text{G-dim}_Q M.$$

2. Upper Gorenstein dimension

In this section $(R, \boldsymbol{m}, k)$ is a local ring and M is a finitely generated R-module.

We say that R has a *Gorenstein deformation*[1] if there exist a local ring Q and a Gorenstein ideal J in Q such that $R = Q/J$. A *Gorenstein quasi-deformation* of R is a diagram of local homomorphisms $R \to R' \leftarrow Q$, with $R \to R'$ a flat extension and $R' \leftarrow Q$ a Gorenstein deformation. We set $M' = M \otimes_R R'$ and define the *upper Gorenstein dimension* of the R-module M by the formula

$$\text{G}^*\text{-dim}_R M = \inf \left\{ \mathrm{pd}_Q M' - \mathrm{pd}_Q R' \;\middle|\; \begin{matrix} R \to R' \leftarrow Q \quad \text{is a} \\ \text{Gorenstein quasi-deformation} \end{matrix} \right\}.$$

Remark 2.1. CM-dimension and G*-dimension are two possible variations on the definition of CI-dimensions. Another way to modify the definition of CM-dimension is to restrict to quasi-deformations $R \to R' \leftarrow Q$ such that the G-perfect Q-module R' satisfies $\mathrm{Ext}^Q_g(R', Q) \cong R'$ for $g = \mathrm{grade}_Q R'$. However it follows from 1.10 that the number defined in such way is precisely the G-dimension of the module. Another variation is to replace in the definition of CI-dimension the CI-deformation by perfect deformation. This one has not been explored yet.

We say that the quasi-deformation $R \to R' \leftarrow Q$ is *embedded* if $\mathrm{edim}\, R = \mathrm{edim}\, Q$. The definition set for G*-dimension can be restricted to the set of embedded Gorenstein quasi-deformations.

Proposition 2.2. *There is an equality*

$$\text{G}^*\text{-dim}_R M = \inf \left\{ \mathrm{pd}_Q M' - \mathrm{pd}_Q R' \;\middle|\; \begin{matrix} R \to R' \leftarrow Q \quad \textit{is an embedded} \\ \textit{Gorenstein quasi-deformation} \end{matrix} \right\}.$$

Proof. Let $R \to R' \leftarrow Q$ be a Gorenstein quasi-deformation and set $M' = M \otimes_R R'$. It is enough to construct an embedded Gorenstein quasi-deformation $R \to R' \leftarrow \overline{Q}$ with the property that $\mathrm{pd}_Q M'$ is finite if and only if $\mathrm{pd}_{\overline{Q}} M'$ is finite.

We may assume $R' = Q/J$, where $(Q, \boldsymbol{n}, k)$ is a local ring and J is a Gorenstein ideal of grade g. If $J \subseteq \boldsymbol{n}^2$, then $\boldsymbol{m}/\boldsymbol{m}^2 \cong \boldsymbol{n}/\boldsymbol{n}^2$ as k-vector spaces and hence

[1] This should not be confused with the G-deformation used in Section 1.

$\operatorname{edim} R' = \operatorname{edim} Q$. Assume $J \not\subseteq \boldsymbol{n}^2$. Since $\operatorname{pd}_Q R'$ is finite we have $g \geq 1$, cf. [11, (1.4.7)], so by prime avoidance there exists a regular element $f \in J \setminus \boldsymbol{n}^2$. Set $\overline{Q} = Q/(f)$. From [4, (3.3.5 (1))] we obtain that $\operatorname{pd}_Q M'$ is finite if and only if $\operatorname{pd}_{\overline{Q}} M'$ is finite. By induction on $\operatorname{rank}_k \left(J/(J \cap \boldsymbol{n}^2)\right)$, it is enough to prove that $\overline{J} = J/(f)$ is a Gorenstein ideal in $\overline{Q}$. Since f is regular, we clearly have $\operatorname{grade}_{\overline{R'}} (\overline{R'}/\overline{J}) = g - 1$, where $\overline{R'} = \overline{Q}/\overline{J} \cong R'$. Furthermore, from [4, (3.3.5 (1))] we get

$$P_{R'}^{\overline{Q}}(t) = \frac{P_{R'}^{Q}(t)}{1+t}.$$

It follows that $\operatorname{pd}_{\overline{Q}} R' = \operatorname{pd}_Q R' - 1 = g - 1$, and $\beta_{g-1}^{\overline{Q}}(\overline{R'}) = \beta_g^Q(R') = 1$, so the ideal $\overline{J}$ of $\overline{R'}$ is Gorenstein. □

The results of this section show that G*-dimension has properties similar to those of the homological dimensions discussed in Section 1.

Lemma 2.3. *If $R \to R' \leftarrow Q$ is a Gorenstein quasi-deformation with $\operatorname{pd}_Q M' < \infty$, then*

$$\operatorname{pd}_Q M' - \operatorname{pd}_Q R' = \operatorname{depth} R - \operatorname{depth}_R M.$$

Proof. From the Auslander-Buchsbaum Equality we obtain

$$\begin{aligned}\operatorname{pd}_Q M' - \operatorname{pd}_Q R' &= (\operatorname{depth} Q - \operatorname{depth}_Q M') - (\operatorname{depth} Q - \operatorname{depth}_Q R') \\ &= \operatorname{depth} R' - \operatorname{depth}_{R'} M'.\end{aligned}$$

Since $R \to R'$ is a flat extension, by [11, (1.2.16)] we have

$$\begin{aligned}\operatorname{depth} R' &= \operatorname{depth} R + \operatorname{depth}_{R'} (R'/\boldsymbol{m}R'); \\ \operatorname{depth}_{R'} M' &= \operatorname{depth}_R M + \operatorname{depth}_{R'} (R'/\boldsymbol{m}R').\end{aligned}$$

Combining the equalities above, we obtain the desired result. □

Proposition 2.4. *If $\text{G}^*\text{-}\dim_R M < \infty$, then $\text{G}^*\text{-}\dim_R M + \operatorname{depth}_R M = \operatorname{depth} R$.*

Proof. By hypothesis there exists a Gorenstein quasi-deformation $R \to R' \leftarrow Q$ with $\operatorname{pd}_Q M' < \infty$, so the result follows from the definition of G_*-dimension and Lemma 2.3. □

Proposition 2.5. *For each $n \geq 0$ there is an equality*

$$\text{G}^*\text{-}\dim_R \operatorname{Syz}_n^R(M) = \max\{\text{G}^*\text{-}\dim_R M - n,\ 0\}.$$

Proof. Let $R \to R' \leftarrow Q$ be a Gorenstein quasi-deformation and set $M' = M \otimes_R R'$. Since $R \to R'$ is flat, we have $\operatorname{Syz}_n^{R'}(M') \cong \operatorname{Syz}_n^R(M) \otimes_R R'$, hence $\operatorname{pd}_Q M' < \infty$ if and only if $\operatorname{pd}_Q \operatorname{Syz}_n^{R'}(M') < \infty$. It follows that $\text{G}^*\text{-}\dim_R M < \infty$ if and only if $\text{G}^*\text{-}\dim_R \operatorname{Syz}_n^R(M) < \infty$, so we can assume $\text{G}^*\text{-}\dim_R M < \infty$. By Proposition 2.4 it is enough to show

$$\operatorname{depth}_R \operatorname{Syz}_n^R(M) = \min\{\operatorname{depth}_R M + n,\ \operatorname{depth} R\} \quad \text{for} \quad 0 \leq n \leq \operatorname{pd}_R M.$$

We induce on n. First, we remark that $\operatorname{depth}_R \operatorname{Syz}_n^R(M) \leq \operatorname{depth} R$ for all $0 \leq n \leq \operatorname{pd}_R M$, cf. Proposition 2.4. For $n = 0$ the equality is trivial. For $n = 1$ we apply the Depth Lemma, cf. e.g. [4, (1.2.6)]. The induction step for $2 \leq n \leq \operatorname{pd}_R M$ follows from the same lemma. □

Proposition 2.6. *There are inequalities*

$$\text{G-dim}_R M \le \text{G}^*\text{-dim}_R M \le \text{CI-dim}_R M$$

with equality to the left of any finite dimension.

Proof. For the first inequality we may assume $\text{G}^*\text{-dim}_R M < \infty$. In this case, there exists a Gorenstein quasi-deformation $R \to R' \leftarrow Q$ with $\text{pd}_Q M' < \infty$, where $M' = M \otimes_R R'$. From 1.9 and 1.10 we conclude that $\text{G-dim}_{R'} M'$ is finite. Since $R \to R'$ is flat, this implies $\text{G-dim}_R M < \infty$, cf. 1.8. The equality $\text{G-dim}_R M = \text{G}^*\text{-dim}_R M$ follows from Proposition 2.4, and 1.5 with H=G.

Since any CI-deformation is a Gorenstein deformation, the second inequality is clear. When $\text{CI-dim}_R M < \infty$, the equality $\text{G}^*\text{-dim}_R M = \text{CI-dim}_R M$, follows from Proposition 2.4, and 1.5 with H=CI. □

Proposition 2.7. *The following conditions are equivalent:*

(i) *The ring R is Gorenstein.*
(ii) $\text{G}^*\text{-dim}_R M < \infty$ *for every R-module M.*
(iii) $\text{G}^*\text{-dim}_R M = 0$ *for every R-module M with* $\text{depth}_R M \ge \text{depth}\, R$.
(iv) $\text{G}^*\text{-dim}_R k = \text{depth}\, R$.
(v) $\text{G}^*\text{-dim}_R k < \infty$.

Proof. (i) $\Longrightarrow$ (ii) Let $\widehat{R}$ be the completion of R with respect to the maximal ideal. Since R is Gorenstein, so is $\widehat{R}$. By Cohen's Structure Theorem $\widehat{R}$ is isomorphic to Q/J, where Q is a regular ring. Since $\widehat{R}$ is Gorenstein and Q is regular, J is perfect hence J is Gorenstein, cf. [11, p. 120]; thus $R \to \widehat{R} \leftarrow Q$ is a Gorenstein quasi-deformation. Since Q is regular, $\text{pd}_Q \widehat{M}$ is finite so $\text{G}^*\text{-dim}_R M$ is finite.

(ii) $\Longrightarrow$ (iii) follows applying Proposition 2.4 to the R-module M.

(iii) $\Longrightarrow$ (iv) by [11, (1.3.7)] we have

$$\text{depth}_R \text{Syz}_n^R(k) \ge \min(n, \text{depth}\, R).$$

In particular, if we choose $n \ge \text{depth}\, R$ we get $\text{G}^*\text{-dim}_R \text{Syz}_n^R(k) = 0$. Thus, by Proposition 2.5, $\text{G}^*\text{-dim}_R k = \text{depth}\, R$.

(iv) $\Longrightarrow$ (v) is trivial.

(v) $\Longrightarrow$ (i) follows from Proposition 2.6, and from 1.7 with H=G. □

Proposition 2.8. *If $R \to R'$ is a local flat extension and $M' = M \otimes_R R'$, then*

$$\text{G}^*\text{-dim}_R M \le \text{G}^*\text{-dim}_{R'} M'$$

with equality when $\text{G}^*\text{-dim}_{R'} M'$ *is finite.*

Proof. We may assume $\text{G}^*\text{-dim}_{R'} M' < \infty$. Let $R' \to R'' \leftarrow Q$ be a Gorenstein quasi-deformation with $\text{pd}_Q M'' < \infty$, where $M'' = M' \otimes_{R'} R''$. Since $R \to R'$ and $R' \to R''$ are flat extensions, the homomorphism $R \to R''$ is also flat, so $R \to R'' \leftarrow Q$ is a Gorenstein quasi-deformation with $\text{pd}_Q (M \otimes_R R'') < \infty$. It follows that $\text{G}^*\text{-dim}_R M$ is finite. Using Proposition 2.6, Property 1.8 with H=G and again Proposition 2.6, we obtain equalities

$$\text{G}^*\text{-dim}_R M = \text{G-dim}_R M = \text{G-dim}_{R'} M' = \text{G}^*\text{-dim}_{R'} M'$$

which finish the proof. □

The next lemma, as well as Lemma 2.11, are folklore results. We provide their proofs because we have not been able to find adequate references.

Lemma 2.9. *Let Q be a local ring and J a Gorenstein ideal with* grade $J = g$. *For every prime ideal $\boldsymbol{q}$ containing J, the ideal $J_{\boldsymbol{q}}$ of $Q_{\boldsymbol{q}}$ is Gorenstein of grade g.*

Proof. Since the ideal J is perfect, the inequalities

$$\operatorname{grade} J \leq \operatorname{grade} J_{\boldsymbol{q}} \leq \operatorname{pd}_{Q_{\boldsymbol{q}}}(Q_{\boldsymbol{q}}/J_{\boldsymbol{q}}) \leq \operatorname{pd}_Q(Q/J)$$

become equalities, so $J_{\boldsymbol{q}}$ is perfect. Localizing a minimal free resolution F of Q/J at $\boldsymbol{q}$ we obtain a resolution $F_{\boldsymbol{q}}$ of $Q_{\boldsymbol{q}}/J_{\boldsymbol{q}}$ which has length g. Thus

$$1 \leq \beta_g^{Q_{\boldsymbol{q}}}(Q_{\boldsymbol{q}}/J_{\boldsymbol{q}}) \leq \operatorname{rank}_{Q_{\boldsymbol{q}}}((F_g)_{\boldsymbol{q}}) = \beta_g^Q(Q/J) = 1,$$

so the ideal $J_{\boldsymbol{q}}$ is Gorenstein. □

Proposition 2.10. *For each prime ideal $\boldsymbol{p} \in \operatorname{Supp}_R(M)$ there is an inequality*

$$\operatorname{G}^*\text{-}\dim_{R_{\boldsymbol{p}}} M_{\boldsymbol{p}} \leq \operatorname{G}^*\text{-}\dim_R M.$$

Proof. We may assume that $\operatorname{G}^*\text{-}\dim_R M$ is finite. Let $R \to R' \leftarrow Q$ be a Gorenstein quasi-deformation with $\operatorname{pd}_Q M' < \infty$, where $M' = M \otimes_R R'$. Since $R \to R'$ is a faithfully flat extension of rings, there exists a prime $\boldsymbol{p}'$ in R' lying over $\boldsymbol{p}$. Let $\boldsymbol{q}$ be the inverse image of $\boldsymbol{p}'$ in Q. The map $R_{\boldsymbol{p}} \to R'_{\boldsymbol{p}'}$ is flat, and by Lemma 2.9 the map $R'_{\boldsymbol{p}'} \leftarrow Q_{\boldsymbol{q}}$ is a Gorenstein deformation, so we see that $R_{\boldsymbol{p}} \to R'_{\boldsymbol{p}'} \leftarrow Q_{\boldsymbol{q}}$ is a Gorenstein quasi-deformation. Lemma 2.9 also gives $\operatorname{pd}_{Q_{\boldsymbol{q}}} R'_{\boldsymbol{q}} = \operatorname{pd}_Q R'$, so we obtain

$$\begin{aligned} \operatorname{G}^*\text{-}\dim_{R_{\boldsymbol{p}}} M_{\boldsymbol{p}} &= \operatorname{pd}_{Q_{\boldsymbol{q}}}(M_{\boldsymbol{p}} \otimes_{R_{\boldsymbol{p}}} R'_{\boldsymbol{p}'}) - \operatorname{pd}_{Q_{\boldsymbol{q}}} R'_{\boldsymbol{q}} \\ &\leq \operatorname{pd}_Q M' - \operatorname{pd}_Q R' \\ &= \operatorname{G}^*\text{-}\dim_R M \end{aligned}$$

Thus, the desired inequality follows. □

Lemma 2.11. *Let Q be a local ring, let $J \subset I$ be ideals of Q, and set $R = Q/J$.*

If J and I/J are Gorenstein ideals in Q and R, respectively, then I is a Gorenstein ideal in Q.

Proof. Set $g = \operatorname{grade}_Q R$ and $h = \operatorname{grade}_R(Q/I)$. The change of rings spectral sequence

$$\mathrm{E}_2^{n,m} = \operatorname{Ext}_R^n(Q/I, \operatorname{Ext}_Q^m(R,k)) \Longrightarrow \operatorname{Ext}_Q^{n+m}(Q/I,k)$$

and the isomorphisms

$$\operatorname{Ext}_Q^i(R,k) = 0 \quad \text{for} \quad i > g \quad \text{and} \quad \operatorname{Ext}_Q^g(R,k) \cong k$$

$$\operatorname{Ext}_R^i(Q/I,k) = 0 \quad \text{for} \quad i > h \quad \text{and} \quad \operatorname{Ext}_R^h(Q/I,k) \cong k$$

yield isomorphisms

$$\operatorname{Ext}_Q^i(Q/I,k) = 0 \quad \text{for} \quad i > g+h \quad \text{and}$$

$$\operatorname{Ext}_Q^{g+h}(Q/I,k) \cong \operatorname{Ext}_R^h(Q/I, \operatorname{Ext}_Q^g(R,k)) \cong \operatorname{Ext}_R^h(Q/I,k) \cong k.$$

Thus, $\operatorname{pd}_Q Q/I = g+h$ and $\beta_Q^{g+h}(Q/I) = 1$. On the other hand, $\operatorname{grade}_Q Q/I = g+h$ by 1.4. Therefore, the ideal I is Gorenstein of grade $g+h$. □

Proposition 2.12. *Let $\boldsymbol{x}$ be a sequence of elements in $\boldsymbol{m}$ which is R-regular and M-regular, and set $\overline{R} = R/(\boldsymbol{x})$ and $\overline{M} = M/(\boldsymbol{x})M$. There is then an inequality*

$$\operatorname{G}^*\text{-}\dim_{\overline{R}} \overline{M} \leq \operatorname{G}^*\text{-}\dim_R M$$

with equality when $\operatorname{G}^*\text{-}\dim_R M$ *is finite.*

Proof. It is enough to prove the proposition for $\boldsymbol{x} = x$ with x an R-regular and M-regular element. We may assume $\mathrm{G}^*\text{-dim}_R M < \infty$ and choose a Gorenstein quasi-deformation $R \to R' \leftarrow Q$ with $\mathrm{pd}_Q M' < \infty$, where $M' = M \otimes_R R'$. Thus, $R' = Q/J$ where J is a Gorenstein ideal.

We construct a Gorenstein quasi-deformation of $\overline{R}$. Choose $y \in Q$ mapping to $x \in R'$. Since x is R-regular, it is also R'-regular due to the flatness of R' as R-module. Set $I = (y) + J$ and note that $I/J = xR'$ is a Gorenstein ideal of R'. By Lemma 2.11 the ideal I is Gorenstein. Set $\overline{R'} = Q/I$, and note that $\overline{R} \to \overline{R'}$ is flat because $R \to R'$ is flat; thus $\overline{R} \to \overline{R'} \leftarrow Q$ is a Gorenstein quasi-deformation of $\overline{R}$.

Next we show $\mathrm{pd}_Q (\overline{M} \otimes_{\overline{R}} \overline{R'}) < \infty$. We have isomorphisms

$$\overline{M} \otimes_{\overline{R}} \overline{R'} \cong \overline{M} \otimes_R \overline{R} \otimes_R R' \cong \overline{M} \otimes_R R'.$$

Since x is M-regular and $R \to R'$ is flat, the sequence $0 \to M \xrightarrow{x} M \to \overline{M} \to 0$ is exact and induces an exact sequence $0 \to M' \xrightarrow{x} M' \to \overline{M} \otimes_R R' \to 0$. From the second sequence we obtain $\overline{M} \otimes_R R' \cong M'/xM'$ and $\mathrm{pd}_Q (M'/xM') = \mathrm{pd}_Q M' + 1$.

The inequality $\mathrm{G}^*\text{-dim}_{\overline{R}} \overline{M} < \infty$ follows. By Proposition 2.4 we have

$$\mathrm{G}^*\text{-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M;$$
$$\mathrm{G}^*\text{-dim}_{\overline{R}} \overline{M} = \operatorname{depth} \overline{R} - \operatorname{depth}_{\overline{R}} \overline{M}.$$

Since x is R-regular and M-regular we also have

$$\operatorname{depth} R = \operatorname{depth} \overline{R} + 1;$$
$$\operatorname{depth}_R M = \operatorname{depth}_{\overline{R}} \overline{M} + 1.$$

The equalities above yield $\mathrm{G}^*\text{-dim}_R M = \mathrm{G}^*\text{-dim}_{\overline{R}} \overline{M}$. □

3. Change of rings

In this section we study the behavior of homological dimensions under certain types of change of rings.

Theorem 3.1. *Let A be a local ring, I an ideal of A, set $B = A/I$, and let $\beta\colon A \to B$ denote the canonical projection. Let $\alpha\colon A \to Q$ be a local homomorphism, set $J = IQ$ and $R = Q/J$, and consider the commutative diagram*

$$\begin{array}{ccccc}
A & \xrightarrow{\ \alpha\ } & A \otimes_A Q & = & Q \\
{\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta \otimes_A Q} & & \downarrow{\scriptstyle\gamma} \\
B & \xrightarrow{\ B \otimes_A \alpha\ } & B \otimes_A Q & = & R
\end{array}$$

Let L be a finitely generated B-module and assume the following hold

$$\mathrm{grade}_Q J \geq \mathrm{grade}_A I, \quad \operatorname{depth}_B L \geq \operatorname{depth} B, \quad \mathrm{pd}_A L < \infty = \mathrm{pd}_B L.$$

The ideal J of Q and the R-module $M = L \otimes_B R$ then have the following properties.

(0) *If I is perfect, then so is J and the module M is perfect over Q with*

$$\mathrm{grade}_Q M = \mathrm{grade}_Q R;$$
$$P^R_M(t) = P^B_L(t);$$
$$0 = \text{CM-dim}_R M < \mathrm{pd}_R M = \infty.$$

(1) *If I is perfect, Golod and not principal, then so is J and*

$$0 = \text{CM-dim}_R M < \text{G-dim}_R M = \infty.$$

(2) *If I is Gorenstein, then so is J and*

$$0 = \mathrm{G}^*\text{-}\dim_R M < \operatorname{pd}_R M = \infty.$$

If, furthermore, $\operatorname{cx}_B L = \infty$, then

$$0 = \mathrm{G}^*\text{-}\dim_R M < \mathrm{CI}_*\text{-}\dim_R M = \infty.$$

(3) *If I is complete intersection, then so is J and*

$$0 = \mathrm{CI}\text{-}\dim_R M < \operatorname{pd}_R M = \infty.$$

In order to prove Theorem 3.1 we need some preparation.

Let $\gamma\colon Q \to R$ be a surjective homomorphism of local rings, and let k denotes the residue field of Q and R.

The homomorphism γ is called *Golod* if

$$P_k^R(t) = \frac{P_k^Q(t)}{1 - t(P_R^Q(t) - 1)}.$$

The homomorphism γ is called *small* if it induces an injective map

$$\operatorname{Tor}_*^\gamma(k,k)\colon \operatorname{Tor}_*^Q(k,k) \to \operatorname{Tor}_*^R(k,k).$$

3.2. If γ is Golod, then it is small, c.f. [2, (3.5)].

An ideal J of Q is called *Golod* (respectively, *small*) if the natural projection $Q \to Q/J$, is Golod (respectively, small).

The ring R is called *Golod* if the canonical map $Q \to \widehat{R}$ is Golod, where $\widehat{R} = Q/J$ is a Cohen presentation with regular ring Q and $\operatorname{edim} Q = \operatorname{edim} \widehat{R}$.

An R-module M is called *inert by γ* if

$$P_k^R(t) P_M^Q(t) = P_M^R(t) P_k^Q(t).$$

The following results are proved by Lescot [16, (6.1)(b); Lemma, p. 43].

3.3. If γ is small and M is an R-module for which the induced map

$$\operatorname{Tor}_*^\gamma(M,k)\colon \operatorname{Tor}_*^Q(M,k) \to \operatorname{Tor}_*^R(M,k)$$

is injective, then M is inert by γ.

3.4. Let $N = \operatorname{Syz}_1^R(M)$. If γ is Golod and $\operatorname{Tor}_i^\gamma(M,k)$ is injective for all $i \geq p$, then $\operatorname{Tor}_i^\gamma(N,k)$ is injective for all $i \geq p-1$.

By [9, (3.5)(2)], if R is a Golod ring and not a hypersurface, then every R-module M with $\mathrm{G}\text{-}\dim_R M < \infty$ has $\operatorname{pd}_R M < \infty$. The idea of that proof, yields the more general result below.

Proposition 3.5. *Let Q be a local ring, J a Golod ideal of Q with $1 < \operatorname{pd}_Q J < \infty$ and set $R = Q/J$. Let M be an R-module with $\operatorname{pd}_Q M = g < \infty$.*

(1) *If $\operatorname{pd}_R M = \infty$, then $\beta_{n+1}^R(M) > \beta_n^R(M)$ for all $n > g$.*
(2) *If $\mathrm{G}\text{-}\dim_R M < \infty$, then $\operatorname{pd}_R M < \infty$.*

Proof. (1) Let γ denote the canonical map $Q \to R$ and set $N = \operatorname{Syz}_{g+1}^R(M)$. Since $\operatorname{Tor}_i^Q(M,k) = 0$ for $i > g$, we can apply 3.4 inductively to obtain the injectivity of $\operatorname{Tor}_*^\gamma(N,k)$. From 3.2 and 3.3 we conclude that N is inert by γ. We now have

$$P_k^R(t) P_N^Q(t) = P_N^R(t) P_k^Q(t) \quad \text{and} \quad P_k^R(t) = \frac{P_k^Q(t)}{1 - t(P_R^Q(t) - 1)}.$$

Combining these equalities we obtain

$$P_N^R(t) = \frac{P_N^Q(t)}{1 - t(P_R^Q(t) - 1)}.$$

Since $\operatorname{pd}_Q R$ and $\operatorname{pd}_Q M$ are finite, so is $\operatorname{pd}_Q N$. The proof of [16, (6.5)] shows that the last equality and the finiteness of $\operatorname{pd}_Q R$ and $\operatorname{pd}_Q N$ imply

$$\beta_{i+1}^R(N) > \beta_i^R(N) \quad \text{for all} \quad i \geq 0.$$

Since $\beta_n^R(M) = \beta_{n-g-1}^R(N)$ for all $n > g$, we obtain the desired conclusion.

(2) We may assume $\operatorname{G-dim}_R M = 0$. Suppose that $\operatorname{pd}_R M$ is infinite; by [9, (8.4)] there exists an exact sequence of non-zero free modules

$$F = \quad \cdots \longrightarrow F_{n+1} \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \longrightarrow \cdots$$

with $\operatorname{rank} F_n = \beta_n^R(M)$ for $n > 0$ and $\partial_n(F_n) \subseteq \boldsymbol{m}F_{n-1}$ for all $n \in \mathbb{Z}$. Set $r = \operatorname{rank} F_0$, $i = -(1 + r + \operatorname{depth} Q)$ and $N = \operatorname{Coker}(\partial_i)$. Since $\operatorname{pd}_R N$ is infinite, from (1), we obtain $\beta_{n-i+1}^R(N) > \beta_{n-i}^R(N)$ for $n \geq -r$. Since $\operatorname{rank} F_n = \beta_{n-i}^R(N)$ for $n \geq -r$, we obtain

$$r = \operatorname{rank} F_0 > \operatorname{rank} F_{-1} > \operatorname{rank} F_{-2} > \cdots > \operatorname{rank} F_{-r} > 0.$$

This is impossible, so we conclude that $\operatorname{pd}_R M$ is finite. □

The next result is proved in [2, (5.6)].

3.6. Under the hypotheses of Theorem 3.1 the following hold:

(1) If I is small, then so is J.
(2) If I is Golod, then so is J.

For the proof of (1), (2) and (3) bellow, see [12, (3.5)]; (4) follows from (3).

3.7. Let A be a local ring, and L a perfect A-module of grade g.
If Q is an A-algebra and $\operatorname{grade}_Q (L \otimes_A Q) \geq g$ then the following hold:

(1) $\operatorname{Tor}_n^A(L, Q) = 0$ for all $n > 0$.
(2) The Q-module $L \otimes_A Q$ is perfect of grade g.
(3) If F is a minimal free resolution of L over A, then $F \otimes_A Q$ is a minimal free resolution of $L \otimes_A Q$ over Q.
(4) There is an equality $P_L^A(t) = P_{L\otimes_A Q}^Q(t)$.

Proof of Theorem 3.1. Set $g = \operatorname{grade}_A I$, and note that in all cases considered in the theorem the ideal I is perfect.

(0) Since $\operatorname{grade}_Q J \geq \operatorname{grade}_A I$, the ideal J is perfect of grade g by 3.7(2).
Using the Auslander-Buchsbaum Equality, we obtain relations

$$\begin{aligned} \operatorname{pd}_A L &= \operatorname{depth} A - \operatorname{depth}_A L \\ &\leq \operatorname{depth} A - \operatorname{depth}_A B \\ &= \operatorname{pd}_A B \\ &= \operatorname{grade}_A B \\ &\leq \operatorname{grade}_A L \end{aligned}$$

which show that the A-module L is perfect of grade g. By 1.4 we have

$$\operatorname{grade}_Q M = \operatorname{grade}_R M + \operatorname{grade}_Q R.$$

Since $\operatorname{grade}_Q R = g$, we obtain $\operatorname{grade}_Q M \geq g$. From 3.7(2), we conclude that M is a perfect module over Q of grade g. From the Auslander-Buchsbaum Equality we then obtain

$$\begin{aligned}\operatorname{depth}_R M &= \operatorname{depth}_Q M \\ &= \operatorname{depth} Q - \operatorname{pd}_Q M \\ &= \operatorname{depth} Q - g \\ &= \operatorname{depth}_Q R \\ &= \operatorname{depth} R.\end{aligned}$$

Applying 3.7(1) to the A-algebra Q and the perfect A-modules B and L we obtain

$$\operatorname{Tor}_n^A(B,Q) = 0 \quad \text{and} \quad \operatorname{Tor}_n^A(L,Q) = 0 \quad \text{for all } n > 0.$$

From the change of rings spectral sequence

$$\mathrm{E}^2_{n,m} = \operatorname{Tor}_n^B(L, \operatorname{Tor}_m^A(B,Q)) \Longrightarrow \operatorname{Tor}_{n+m}^A(L,Q)$$

we get $\operatorname{Tor}_n^B(L,R) \cong \operatorname{Tor}_n^A(L,Q) = 0$ for all $n > 0$. Let G be a minimal free resolution of L as B-module. Since $B \to R$ is a local map, $G \otimes_B R$ is a minimal free resolution of M over R, hence $P_L^B(t) = P_M^R(t)$; in particular $\operatorname{pd}_R M = \infty$.

By 1.9, the module M has finite G-dimension over Q and the ideal J is G-perfect. Thus, $R \leftarrow Q$ is a G-deformation which gives $\text{CM-dim}_R M < \infty$. We obtain $\text{CM-dim}_R M = 0$ from 1.5 with H=CM.

(1) When I is Golod, J is also Golod by 3.6. Since I is perfect, not principal and $\operatorname{grade} I = \operatorname{grade} J$, the ideal J is not principal. Assuming $\text{G-dim}_R M < \infty$, from Lemma 3.5, we obtain $\infty = \operatorname{pd}_R M < \infty$, a contradiction.

In the remaining parts (2) and (3) of the theorem we need to show

$$0 = \text{H-dim}_R M < \operatorname{pd}_R M = \infty$$

for H=G and CI. By 1.9 and (0), it is enough to prove $\text{H-dim}_R M < \infty$.

(2) Suppose that I is Gorenstein. The ideal J is perfect of grade g by (0), and we have $\beta_g^A(A/I) = \beta_g^Q(Q/J) = 1$ by 3.7(4). Thus, the ideal J is Gorenstein. The Gorenstein deformation $R \leftarrow Q$ satisfies $\operatorname{pd}_Q M < \infty$, hence $\text{G}^*\text{-dim}_R M < \infty$.

If $\operatorname{cx}_B L$ is infinite, then $\operatorname{cx}_R M$ is infinite because $P_L^B(t) = P_M^R(t)$ by (0). But $\operatorname{cx}_R M = \infty$ implies $\text{CI}_*\text{-dim}_R M = \infty$ cf. 1.1. Therefore, $\text{G}^*\text{-dim}_R M < \text{CI}_*\text{-dim}_R M$.

(3) Assume that I is generated by an A-regular sequence $(x_1, x_2, ..., x_g)$. Since $J = (x_1, x_2, ..., x_g)Q$ has grade $\geq g$, the sequence $(x_1, x_2, ..., x_g)$ is Q-regular, so the ideal J is complete intersection. The CI-deformation $R \leftarrow Q$ satisfies $\operatorname{pd}_Q M < \infty$, hence $\text{CI-dim}_R M < \infty$. □

4. Infinite homological dimensions

In this section we prove the first three parts of the Main Theorem stated in the introduction.

Throughout the section, $(Q, \boldsymbol{n}, k)$ denotes a local ring and J denotes an ideal in Q contained in $\boldsymbol{n}^2$. We set $R = Q/J$ and $p = \operatorname{char} k$.

Part 1. *If J is perfect, not complete intersection, with* $\operatorname{grade} J = 2$, *then there exist R-modules M with*

$$0 = \text{CM-dim}_R M < \text{G-dim}_R M = \infty.$$

Proof. By the Hilbert-Burch Theorem [11, (1.4.17)], there exists a matrix

$$\boldsymbol{x} = (x_{ij})_{\substack{i=1,\dots,n+1\\ j=1,\dots,n}}$$

with entries in $\boldsymbol{n}$ such that $J = I_n(\boldsymbol{x})$, where $I_n(\boldsymbol{x})$ denotes the ideal generated by the $n \times n$ minors of the matrix $\boldsymbol{x}$. Let

$$X = \{X_{ij} \mid i = 1, \dots, n+1,\ j = 1, \dots, n\}$$

be a set of indeterminates over $\mathbb{Z}$, and set $A = \mathbb{Z}[X]_{(p,X)}$. Let $\alpha\colon A \to Q$ be the local homomorphism given by

$$\alpha(X_{ij}) = x_{ij} \quad \text{for} \quad i = 1, \dots, n+1 \quad \text{and} \quad j = 1, \dots, n.$$

Consider the $(n+1) \times n$ matrix

$$\boldsymbol{X} = (X_{ij})_{\substack{i=1,\dots,n+1\\ j=1,\dots,n}}$$

and $I = I_n(\boldsymbol{X})$. The ideal I is perfect of grade 2 cf. [11, (1.4.17)], and $IQ = J$.

Set $B = A/I$. Since B is not regular, there exist non-free maximal Cohen-Macaulay B-modules L, cf. 1.7. Since $\operatorname{grade} I = 2$ and B is not complete intersection, I is Golod cf. [4, (5.3.4)]. The conclusion follows from Theorem 3.1(3). □

Part 2. *If J is Gorenstein, not complete intersection, with* $\operatorname{grade} J = 3$, *then there exist R-modules M with*

$$0 = \text{G}^*\text{-}\dim_R M < \text{CI}_*\text{-}\dim_R M = \infty.$$

Proof. By the Buchsbaum-Eisenbud Theorem [11, (3.4.1)(b)], for some $r \geq 2$, there exists an alternating matrix

$$\boldsymbol{x} = (x_{ij})_{i,j=1,\dots,2r+1}$$

with entries in $\boldsymbol{n}$, such that J is generated by the Pfaffians of the matrices obtained by deleting the i'th line and column of $\boldsymbol{x}$ for each $i = 1, \dots, 2r+1$. Let

$$X = \{X_{ij} \mid 1 \leq i < j \leq 2r+1\}$$

be a set of indeterminates over $\mathbb{Z}$, and set $A = \mathbb{Z}[X]_{(p,X)}$. Let $\alpha\colon A \to Q$ be the local homomorphism given by

$$\alpha(X_{ij}) = x_{ij} \quad \text{for} \quad 1 \leq i < j \leq 2r+1.$$

Consider the $(2r+1) \times (2r+1)$ matrix

$$\boldsymbol{X} = (X_{ij})_{i,j=1,\dots,2r+1}$$

with $X_{ii} = 0$ and $X_{ji} = -X_{ij}$ for $j > i$. Let I be the ideal generated by the Pfaffians of the matrices obtained by deleting the i'th line and column of $\boldsymbol{X}$ for each $i = 1, \dots, 2r+1$. The ideal I is Gorenstein of grade 3 cf. [11, (3.4.1)(a)], and $IQ = J$.

Set $B = A/I$. Since $\operatorname{pd}_A B = 3$, by the Auslander-Buchsbaum Equality we have

$$\operatorname{depth} B = \operatorname{depth} A - 3.$$

By construction, A is a regular and an embedded deformation of B, so $\operatorname{codepth} B = 3$. The ring B is not complete intersection, so by 1.7, there exist maximal Cohen-Macaulay B-modules L with $\text{CI-}\dim_B L = \infty$. Applying [3, (1.6.IV)] to the B-module L, we obtain $\operatorname{cx}_B L = \infty$. The conclusion follows from Theorem 3.1(2). □

Part 3. *If J is complete intersection with* grade $J \geq 1$*, then there exist R-modules M with*

$$0 = \text{CI-dim}_R M < \text{pd}_R M = \infty.$$

Proof. Let $(a_1, \ldots, a_g)$ be a regular sequence generating J, let $(x_1, \ldots, x_n)$ be a minimal system of generators of $\boldsymbol{n}$ and note that $g \leq n$. Since $J \subseteq \boldsymbol{n}^2$, we can write

$$a_h = \sum_{i=1}^{n} y_{hi} x_i \quad \text{for} \quad h = 1, \ldots, g$$

where $y_{hi} \in \boldsymbol{n}$ for $i = 1, \ldots, n$ and $h = 1, \ldots, g$. Consider the following sets of indeterminates over $\mathbb{Z}$:

$$X = \{X_1, \ldots, X_n\} \quad \text{and} \quad Y = \{Y_{hi} \mid 1 \leq h = 1, \ldots, g, \ i = 1, \ldots, n\}.$$

Set $A = \mathbb{Z}[X, Y]_{(p,X,Y)}$ and let $\alpha \colon A \to Q$ be the local homomorphism given by

$$\alpha(X_i) = x_i \quad \text{and} \quad \alpha(Y_{hi}) = y_{hi}, \quad \text{for} \quad h = 1, \ldots, g \quad \text{and} \quad i = 1, \ldots, n.$$

Let I be the ideal of A generated by the elements

$$f_h = \sum_{i=1}^{n} Y_{hi} X_i \quad \text{for} \quad h = 1, \ldots, \ g.$$

Consider two sequences of elements in A. The first one, denoted $\boldsymbol{y}$, is given by the elements of the set

$$\{Y_{hi} \mid h = 1, \ldots, g, \ i = 1, \ldots, n, \ \text{and } i \neq h\}$$

taken in an arbitrary order. The second one is

$$\boldsymbol{f} = (f_1, \ldots, f_g).$$

It is clear that $\boldsymbol{y}$ is A-regular. Since

$$A/(\boldsymbol{y}) \cong \mathbb{F}_p[X_1, \ldots, X_n, Y_{11}, \ldots, Y_{gg}] \quad \text{and}$$
$$(\boldsymbol{f}) + (\boldsymbol{y}) = (X_1 Y_{11}, \ldots, X_g Y_{gg}) + (\boldsymbol{y}),$$

it follows that $\boldsymbol{f}$ is $A/(\boldsymbol{y})$-regular. Since A is local and the sequence $(\boldsymbol{y}, \boldsymbol{f})$ is A-regular, we see that $(\boldsymbol{f}, \boldsymbol{y})$ is A-regular cf. [11, (1.1.6)]. In particular, $\boldsymbol{f}$ is A-regular, hence the ideal I is complete intersection.

Set B=A/I. Since B is not regular, so there exists a non-free maximal Cohen-Maculay B-module L, cf. 1.7 with H=P and H=CM. Since $IQ = J$, the conclusion follows from Theorem 3.1(1). ☐

5. Infinite CI-dimension

In this section we prove the last two parts of the Main Theorem. For this we need more preparation.

For each local ring B with residue field l there exists a graded Lie algebra over l, denoted $\pi^*(B)$, such that the universal enveloping algebra of $\pi^*(B)$ is equal to the algebra $\text{Ext}^*_B(l, l)$ with Yoneda products. For details of the construction of $\pi^*(B)$ see e.g. [4, §10].

The following results are from [10, (5.3)] and [8, (7.3)(1)].

5.1. Let (B, l) be a local ring and let L be a finite B-module. If $\text{CI-dim}_B L < \infty$ then the following hold.

(1) The left $\text{Ext}^*_B(l, l)$-module $\text{Ext}^*_B(L, l)$ is finite over the l-subalgebra $\mathcal{P}^*$ of $\text{Ext}^*_B(l, l)$, generated by the central elements in $\pi^2(B)$.

(2) If $\operatorname{cx}_R M = 1$, then $\operatorname{Syz}_{n+2}^B(L) \cong \operatorname{Syz}_n^B(L)$ for all $n \gg 0$.

We use two specific rings which admit modules with constant Betti numbers 2, hence of complexity 1. The first ring is a variation of an example of Avramov, Gasharov and Peeva proposed by Löfwall, cf. [7, (2.2)(i)] and the second is an example of Gasharov and Peeva [13, (3.1)].

Example 5.2. Let l be a field, let $X = \{X_1, X_2, X_3, X_4\}$ be a set of indeterminates over l and set $A = l[X]_{(X)}$. Let I be the ideal of A generated by the elements

$$X_1^2,\ X_1X_2 - X_3X_4,\ X_1X_2 - X_4^2,\ X_1X_3 - X_2X_4,$$
$$X_1X_4 - X_2^2,\ X_1X_4 - X_2X_3,\ X_1X_4 - X_3^2$$

and set $B = A/I$.

5.3. Let B be the ring defined in 5.2 and let x_i denote the image of X_i in B for $i = 1, \dots, 4$. Consider the sequence of homomorphisms of free B-modules

$$F = \quad \cdots \xrightarrow{\psi} B^2 \xrightarrow{\varphi} B^2 \xrightarrow{\psi} B^2 \xrightarrow{\varphi} \cdots$$

where

$$\varphi = \begin{pmatrix} x_3 & x_1 \\ x_4 & x_2 \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} x_2 & -x_1 \\ -x_4 & x_3 \end{pmatrix}.$$

Set $L = \operatorname{Coker} \varphi$. By [7, (2.2)(i)] the complex F is exact, so $\beta_B^n(L) = 2$ for all $n \geq 0$. A similar computation to [7, Section 3] shows that the complex F^* is also exact hence $\operatorname{G-dim}_B L = 0$ cf. e.g. [9, (3.1)].

Lemma 5.4. *For the ring B in 5.2 the graded Lie algebra $\pi^*(B)$ contains no non-zero central element of degree 2.*

Proof. By the arguments in [7, p. 4], the fact that B has a module L with constant Betti numbers, cf. Lemma 5.3, implies that the Poincaré series of l over B is given by the formula

$$P_l^B(t) = \frac{1}{1 - 4t + 3t^2}.$$

Writing it as an infinite product

$$P_l^B(t) = \frac{(1+t)^4(1+t^3)^8(1+t^5)^{\varepsilon_5} \dots}{(1-t^2)^7(1-t^4)^{\varepsilon_4} \dots},$$

from [4, (10.2.1)(2)], we obtain the equalities

$$\operatorname{rank}_l \pi^1(B) = 4, \quad \operatorname{rank}_l \pi^2(B) = 7, \quad \operatorname{rank}_l \pi^3(B) = 8.$$

By the arguments of [7, p. 4], we see that the graded Lie algebra $\pi^*(B)$ is generated by elements t_1, t_2, t_3 and t_4 of degree 1 with the following relations:

$$[t_1, t_2] + [t_3, t_4] + t_4^{(2)} = 0,$$
$$[t_1, t_3] + [t_2, t_4] = 0,$$
$$[t_1, t_4] + t_2^{(2)} + [t_2, t_3] + t_3^{(2)} = 0.$$

A direct computation then shows that the following elements of $\pi^2(B)$ are linearly independent, and hence form a basis

$$u_1 = t_1^{(2)},\ u_2 = [t_1, t_2],\ u_3 = [t_1, t_3],\ u_4 = [t_1, t_4],$$
$$u_5 = t_2^{(2)},\ u_6 = [t_2, t_3],\ u_7 = [t_3, t_4].$$

Computing the brackets $[t_i, u_j]$ for $i = 1, \dots, 4$ and $j = 1, \dots, 7$ by using the Jacobi Identity, and using the relations $[t_i, t_i^{(2)}] = 0$ for $i = 1, \dots, 4$ we see that the following elements form a basis for $\pi^3(B)$

$$v_i = [t_1, u_{i+1}] \text{ for } i = 1, \dots, 6,\ v_7 = [t_2, t_4],\ v_8 = [t_2, t_6].$$

Let $u = \sum_{i=1}^{7} a_i u_i$ be a central element of degree 2 in $\pi^*(B)$. The equality $[t_1, u] = 0$ implies $\sum_{i=2}^{7} a_i v_{i-1} = 0$, so $t = a_1 t_1$. The equality $[t_2, u] = 0$ then implies $-a_1 v_1 = 0$, so we conclude that $u = 0$. □

Throughout the section, $(Q, \boldsymbol{n})$ and J denotes an ideal in Q contained in $\boldsymbol{n}^2$. We set $R = Q/J$.

Part 4. *If Q is an algebra over a field l and* depth $Q \geq 4$*, then there exists a perfect ideal J of grade* 4 *and an R-module M with*

$$0 = \text{CI}_*\text{-dim}_R\, M < \text{CI-dim}_R\, M = \infty.$$

Proof. Let A and B be the rings defined in 5.2. The ideal I is perfect of grade 4 because it contains $X_1^3, \dots, X_4^3$. Let $(a_1, \dots, a_4)$ be a regular sequence in $\boldsymbol{n}$, let $\alpha\colon A \to Q$ be the local homomorphism given by

$$\alpha(X_i) = a_i \quad \text{for} \quad i = 1, \dots, 4$$

and set $J = IQ$. The ideal J contains the Q-regular sequence $a_1^3, \dots, a_4^3$, hence grade $J \geq 4$. Applying 3.7(2) we see that the ideal J is perfect of grade 4. Let L be the B-module with G-dim$_B\, L = 0$, from 5.3 and set $M = L \otimes_B R$. Since α is flat cf. e.g. [18, Exercise 22.2], we obtain G-dim$_R\, M = 0$ cf. 1.8. By Theorem 3.1(1) we have $P_M^R(t) = P_L^B(t) = 2/(1-t)$; in particular, $\text{cx}_R\, M = 1$, hence $\text{CI}_*\text{-dim}_R\, M = 0$.

It remains to prove that CI-dim$_R\, M$ is infinite. Since α is local flat homomorphism, so is $B \otimes_A \alpha$, hence applying 1.8 with H=CI we get

$$\text{CI-dim}_B\, L \leq \text{CI-dim}_R\, M.$$

Thus, it is enough to show CI-dim$_B\, L = \infty$. If we suppose that CI-dim$_B\, L < \infty$, then Lemma 5.4 and 5.1(1) imply $\text{Ext}_B^n(L, l) = 0$ for all $n \gg 0$. This is imposible, since $\text{Ext}_B^n(L, l) \cong l^2$ for all $n \geq 0$. □

5.5. Let l be a field and let a be a non-zero element of l, let $X = \{X_1, X_2, X_3, X_4, X_5\}$ be a set of indeterminates over l, and set $A = l[X]_{(X)}$. Let I be the ideal of A generated by the elements

$$aX_1X_3 + X_2X_3,\ X_1X_4 + X_2X_4,\ X_3^2 - X_2X_5 + aX_1X_5,$$
$$X_4^2 - X_2X_5 + X_1X_5,\ X_1^2,\ X_2^2,\ X_3X_4,\ X_3X_5,\ X_4X_5,\ X_5^2$$

and set $B = A/I$. This ring is Gorenstein by [13, (3.1)(i)].

5.6. Let B be the ring defined in 5.5 and let x_i denote the image of X_i in B for $i = 1, \dots, 5$. Consider the sequence of homomorphisms of free B-modules

$$F = \quad \cdots \longrightarrow B^2(-n-1) \xrightarrow{\varphi_{n+1}} B^2(-n) \xrightarrow{\varphi_n} B^2(-n+1) \longrightarrow \cdots$$

where

$$\varphi_n = \begin{pmatrix} x_1 & a^n x_3 + x_4 \\ 0 & x_2 \end{pmatrix}.$$

Gasharov and Peeva [13, (3.1)] prove that F is a minimal free resolution of $L = \text{Coker}\,\varphi_0$, and that $\text{Syz}_{n+q}^B(L) \cong \text{Syz}_n^B(L)$ for all $n \gg 0$ if and only if $a^q = 1$.

Part 5. *If Q is an algebra over a field l with* $\operatorname{card}(l) > 3$*, and* $\operatorname{depth} Q \geq 5$*, then there exists a Gorenstein ideal J with* $\operatorname{grade} J = 5$ *and an R-module M with*

$$0 = \mathrm{G}^*\text{-}\dim_R M = \mathrm{CI}_*\text{-}\dim_R M < \mathrm{CI}\text{-}\dim_R M = \infty.$$

Proof. Since $l \neq \{0, \pm 1\}$, there exists an element a of l such that $a^2 \neq 1$. Let B be the Gorenstein ring defined in 5.5. As the ring A is regular, the ideal I is Gorenstein cf. e.g. [11, p.120]. Let $(a_1, \ldots, a_5)$ be a regular sequence in $\boldsymbol{n}$, let $\alpha\colon A \to Q$ be the local homomorphism given by

$$\alpha(X_i) = a_i \quad \text{for} \quad i = 1, \ldots, 5$$

and set $J = IQ$. Since I contains the elements $X_1^4, \ldots, X_5^4$, the ideal J contains A-regular sequence $a_1^4, \ldots, a_5^4$, hence $\operatorname{grade} J \geq \operatorname{grade} I = 5$. Let L be the module from 5.6 and set $M = L \otimes_B R$. Applying Theorem 3.1(2) we obtain that the ideal J is Gorenstein of grade 5 and $\mathrm{G}^*\text{-}\dim_R M = 0$. By Theorem 3.1(1) and 5.6 we see that $\operatorname{cx}_R M = 1$, therefore by definition and by Proposition 2.6 we have $\mathrm{CI}_*\text{-}\dim_R M = 0$. It remains to prove $\mathrm{CI}\text{-}\dim_R M = \infty$. Since α is local flat homomorphism, so is $B \otimes_A \alpha$, hence applying 1.8 with H=CI we get $\mathrm{CI}\text{-}\dim_B L \leq \mathrm{CI}\text{-}\dim_R M$. Thus, it is enough to show $\mathrm{CI}\text{-}\dim_B L = \infty$. From 5.6 and the choice of a, $\operatorname{Syz}_{n+2}^B(L)$ is not isomorphic to $\operatorname{Syz}_n^B(L)$ for $n \gg 0$. Therefore $\mathrm{CI}\text{-}\dim_B L$ is infinite by 5.1(2). ☐

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