Quiz (SOLUTIONS)

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Math 2270, Fall 2006

Prove the following fact: Consider a symmetric matrix A. If $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of A with distinct eigenvalues λ_1 and λ_2, then $\vec{v}_1 \cdot \vec{v}_2 = 0$.

We compute $\vec{v}_1^T A \vec{v}_2$ in two ways:

$$\vec{v}_1^T A \vec{v}_2 = \vec{v}_1^T \lambda_2 \vec{v}_2 = \lambda_2 (\vec{v}_1 \cdot \vec{v}_2)$$

$$\vec{v}_1^T A \vec{v}_2 = \vec{v}_1^T A^T \vec{v}_2 = (A\vec{v}_1)^T \vec{v}_2 = (\lambda_1 v_1)^T \vec{v}_2 = \lambda_1 (\vec{v}_1 \cdot \vec{v}_2)$$

$$\lambda_2 (\vec{v}_1 \cdot \vec{v}_2) = \lambda_1 (\vec{v}_1 \cdot \vec{v}_2) \iff \underbrace{(\lambda_2 - \lambda_1)}_{\neq 0} (\vec{v}_1 \cdot \vec{v}_2) = 0$$

$$\Rightarrow \vec{v}_1 \cdot \vec{v}_2 = 0 \Rightarrow \vec{v}_1 \perp \vec{v}_2 \quad \blacksquare$$

Problem: For the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ find an orthogonal matrix S and a diagonal matrix D such that $S^{-1}AS = D$.

$$A - \lambda I_3 = \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} \Rightarrow f_A(\lambda) = \lambda^2(1-\lambda) - (1-\lambda)$$

$$= (1-\lambda)(\lambda^2 - 1)$$

$$= (1-\lambda)(\lambda-1)(\lambda+1)$$

$\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$ are eigenvalues.

$$E_1 = \ker \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

$$E_{-1} = \ker \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$$

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{v}_1$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\,\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 0 \cdot \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\vec{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \; ; \; D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A = SDS^{-1} = SDS^{T}.$$