Final Exam **Spring 2005** **MATH 2270-01**

Instructor: Oana Veliche
Time: 2 hours

SOLUTIONS

NAME:______________________________

ID#:______________________________

INSTRUCTIONS
(1) Fill in your name and your student ID number.
(2) Justify all your assertions.
(3) No books, notes or calculators may be used.

Page #	2	3	4	5	6	7	8	9	10	11	12	13	14	Total
Max. # points	20	15	15	25	20	10	15	10	15	10	10	20	15	200
# Points														

Problem 1. Consider the following symmetric matrix:

$$B = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}.$$

(a)(5 points) Find the eigenvalues of B.

$$f_\lambda(B) = \lambda^2 - 10\lambda + (25-9)$$
$$= \lambda^2 - 10\lambda + 16$$
$$= (\lambda-2)(\lambda-8)$$

$$\lambda_1 = 8$$
$$\lambda_2 = 2$$

(b)(10 points) Find an eigenbasis for the matrix B.

$$E_8 = \ker \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$E_2 = \ker \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

(c)(5 points) Find an orthonormal eigenbasis for the matrix B.

$$\mathcal{V} = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

(c)(10 points) Diagonalize the matrix B. Check your answer!

$$B = S D S^{-1} \qquad S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det S = -2$$

$$S^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 8 & -2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & +\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4+1 & +4-1 \\ 4-1 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

(d)(5 points) Using (c) diagonalize the matrix $C = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$.

$$C = B - 2I_2$$

$$C = S D S^{-1} - 2 S S^{-1}$$

$$= S (D - 2I_2) S^{-1}$$

$$= S \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix} S^{-1}, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Problem 2. Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $L(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}.$$

(a)(5 points) Find the singular values of the matrix A. Hint: remark that $A^TA = B$, from Problem 1.

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{8} = 2\sqrt{2}$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{2}$$

(b)(10 points) Find a singular values decomposition of the matrix A.

$$A = U\Sigma V^T \qquad \Sigma = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$A \cdot \vec{v}_1 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} \Rightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = V^T$$

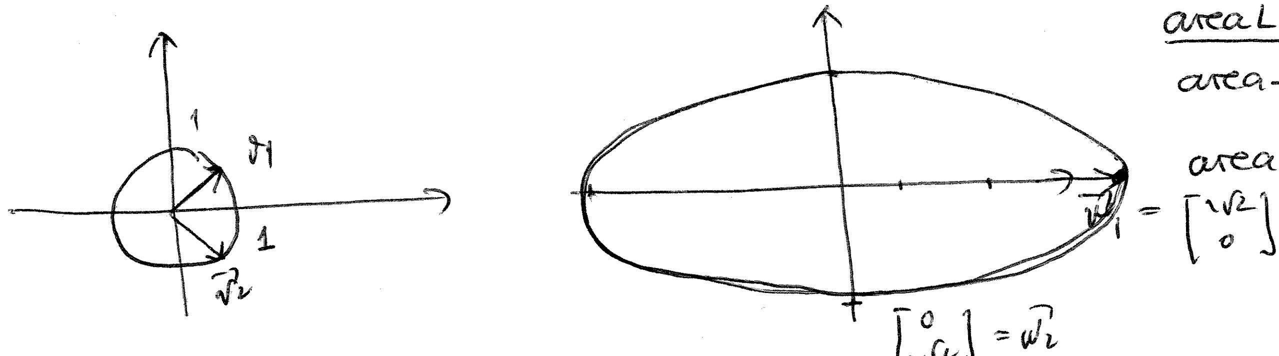
$$A \cdot \vec{v}_2 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix} \Rightarrow \vec{u}_2 = \frac{1}{\sqrt{2}} A\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

(c)(10 points) Draw the image of the unit circle Ω under the transformation L and find the area of $L(\Omega)$.

$$L(\cos t\, \vec{v}_1 + \sin t\, \vec{v}_2) = \cos t\, L(\vec{v}_1) + \sin t\, L(\vec{v}_2)$$
$$= \cos t\, \vec{w}_1 + \sin t\, \vec{w}_2$$



$$\frac{\text{area } L(\Omega)}{\text{area } \Omega} = |\det(A)|$$

$$\text{area } L(\Omega) = \pi \cdot 4$$

Problem 3. (a)(10 points) Let B be the matrix from Problem 1. Show that the quadric $q(\vec{x}) = \vec{x} \cdot B\vec{x}$ is not indefinite (remark that $B = A^T A$ with A from Problem 2).

$q(\vec{x}) = \lambda_1 c_1^2 + \lambda_2 c_2^2$ with $\lambda_1 = 8$, $\lambda_2 = 2$

$\Rightarrow q(\vec{x}) \geq 0$ for all $\vec{x}$ in $\mathbb{R}^2$, $\Rightarrow$ $q(\vec{x})$ does not take negative values. $\Rightarrow q$ is not indefinite.

(b)(5 points) Explain why a quadric $q\colon \mathbb{R}^m \to \mathbb{R}$ given by

$$q(\vec{x}) = \vec{x} \cdot (A^T A)\vec{x}$$

is never indefinite for any choice of an $n \times m$ matrix A?

The eigenvalues of $A^T A$ are always positive.

$\Rightarrow q(\vec{x}) = \lambda_1 c_1^2 + \dots + \lambda_n c_n^2 \geq 0$

$\Rightarrow$ q does not take negative values

$\Rightarrow$ q is not indefinite.

Problem 4. Let $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -k^2 \\ \frac{1}{4} & \frac{\sqrt{3}}{2} \end{bmatrix}$ where k is a real number.

(a)(10 points) Find all k such that $\vec{0}$ is a stable equilibrium for the dynamical system $\vec{x}(t+1) = A\vec{x}(t)$.

$\vec{0}$ is a stable equilibrium iff $|\lambda_1|$ and $|\lambda_2| < 1$, where λ_1 and λ_2 are the eigenvalues of A.

$$f_\lambda(A) = \left(\frac{\sqrt{3}}{2} - \lambda\right)^2 + \frac{k^2}{4} = 0 \Rightarrow \frac{\sqrt{3}}{2} - \lambda = \pm i\frac{k}{2}$$

$$\Rightarrow \lambda_{1,2} = \frac{\sqrt{3}}{2} \pm i\cdot\frac{k}{2} \qquad |\lambda_{1,2}| = \sqrt{\frac{3}{4} + \frac{k^2}{4}} < 1$$

$$\Leftrightarrow \frac{k^2}{4} < \frac{1}{4} \Leftrightarrow k^2 < 1 \Leftrightarrow |k| < 1$$

(b)(10 points) Find the real closed formula for the trajectory $\vec{x}(t+1) = A\vec{x}(t)$ with $k = 1$ and $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$A^t = r^t S \begin{bmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{bmatrix} S^{-1} \vec{x}_0 \qquad S = [\vec{w}, \vec{v}]$$

$$E_{\frac{\sqrt{3}}{2} + i\frac{1}{2}} = \ker \begin{bmatrix} -\frac{i}{2} & -1 \\ \frac{1}{4} & -\frac{i}{2} \end{bmatrix} = \text{span}\left(\begin{bmatrix} 1 \\ \frac{i}{2} \end{bmatrix}\right) \Rightarrow \begin{bmatrix} 1 \\ \frac{i}{2} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{v}} + i \underbrace{\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}}_{\vec{w}}$$

$$\frac{\sqrt{3}}{2} + i\frac{1}{2} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right), \quad r = 1 \qquad S = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{6}t\right) & -\sin\left(\frac{\pi}{6}t\right) \\ \sin\left(\frac{\pi}{6}t\right) & \cos\left(\frac{\pi}{6}t\right) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad S^{-1} = -2\begin{bmatrix} 0 & -\frac{1}{2} \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sin\left(\frac{\pi}{6}t\right) & \cos\left(\frac{\pi}{6}t\right) \\ 2\cos\left(\frac{\pi}{6}t\right) & -2\sin\left(\frac{\pi}{6}t\right) \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \sin\left(\frac{\pi}{6}t\right) + \frac{1}{2}\cos\left(\frac{\pi}{6}t\right) \\ 2\cos\left(\frac{\pi}{6}t\right) - \sin\left(\frac{\pi}{6}t\right) \end{bmatrix}$$

Problem 5.(10 points) Using Cramer's rule solve the following system:

$$\left|\begin{array}{ccccc} x & + & y & = & 2 \\ x & + & z & = & 0 \\ y & + & z & = & 0 \end{array}\right|.$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \implies \det(A) = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= -1 - 1 = -2$$

$$x = \frac{\det \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}{-2} = \frac{2 \cdot \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}{-2} = \underline{1}$$

$$y = \frac{\det \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}}{-2} = \frac{-2 \cdot \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}{-2} = \underline{1}$$

$$z = \frac{\det \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}{-2} = \frac{2 \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{-2} = -1$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Problem 6. Let P_1 be the space of all polynomials of degree ≤ 1 and consider the function:

$$<,>: P_1 \times P_1 \to \mathbb{R}$$

given by the formula:

$$< f, g >= \frac{1}{2}[f(0)g(0) + f(1)g(1)] + k,$$

for some k in $\mathbb{R}$.

(a)(5 points) Show that $<,>$ is not an inner product if $k \neq 0$.

$$\langle 0, 0 \rangle = \frac{1}{2} 0 + k = 0 \Rightarrow k = 0,$$

(b)(10 points) Find an orthonormal basis of P_1 with the inner product

$$< f, g >= \frac{1}{2}[f(0)g(0) + f(1)g(1)].$$

$\mathcal{B} = (1, t)$ basis of P_1 $\Rightarrow$ $\mathcal{U} = (1, 2t-1)$

$$\langle 1, 1 \rangle = \frac{1}{2}(1+1) = 1$$

$$\langle 1, t \rangle = \frac{1}{2}(1 \cdot 0 + 1 \cdot 1) = \frac{1}{2}$$

$$t^\perp = t - \langle 1, t \rangle \cdot 1 = t - \frac{1}{2}$$

$$\|t^\perp\| = \sqrt{\langle t - \tfrac{1}{2}, t - \tfrac{1}{2} \rangle} = \sqrt{\frac{1}{2}\underbrace{\left(\cancel{\tfrac{1}{2} \cdot \tfrac{1}{2}} + \tfrac{1}{2}\tfrac{1}{2}\right)}_{\frac{1}{2}}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\frac{t^\perp}{\|t^\perp\|} = 2\left(t - \frac{1}{2}\right) = 2t - 1$$

Problem 7. Let A be an $n \times n$ matrix with all the entries integers.

(a)(5 points) Show that if $|\det(A)| = 1$, then A^{-1} has also all the entries integers.

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

$$\left.\begin{array}{l} \operatorname{adj}(A) \text{ has integer entries.} \\ \det(A) = \pm 1 \end{array}\right\} \Rightarrow A^{-1} \text{ has also integer entries.}$$

(b)(5 points) Show that if $A^2 + A + I_n = 0$, then $|\det(A)| = 1$.

$$A^2 + A + I_n = 0$$

$$A(A + I_n) = -I_n$$

$$\det(A) \cdot \det(A + I_n) = \det(-I_n) = (-1)^n \det I_n$$
$$= (-1)^n$$

$$\Rightarrow \underbrace{|\det(A)|}_{\text{integer}} \cdot \underbrace{|\det(A + I_n)|}_{\text{integer.}} = 1$$

$$\Rightarrow |\det(A)| = 1.$$

Problem 8. Let A be an 3×5 matrix.

(a)(5 points) Show that if $\vec{v}$ is a vector in $\operatorname{im}(A) \cap \ker(A^T)$, then $\vec{v} = \vec{0}$.

$$\left.\begin{array}{l} \ker(A^T) = \operatorname{Im}(A)^{\perp} \\ V \cap V^{\perp} = \{\vec{0}\} \end{array}\right\} \Rightarrow \vec{v} = \vec{0}$$

$\boxed{\text{or}}$

$$\left.\begin{array}{l} A\vec{w} = \vec{v} \\ A^T\vec{v} = \vec{0} \end{array}\right\} \Rightarrow A^TA\vec{w} = 0 \Rightarrow \vec{w} \in \ker(A^TA)$$

$$\ker(A^TA) = \ker A$$

$$\Rightarrow A\vec{w} = \vec{0} = \vec{v}.$$

(b)(10 points) Compute nullity(A^T) if rank$(A) = 2$.

A 3×5 matrix $\Rightarrow$ A^T 5×3 matrix. 2pts

5pts nullity(A^T) + rank$(A^T) = 3$

rank(A^T) = rank(A) = 2 2pt

$\Rightarrow$ nullity$(A^T) = 1$

1pt

Problem 9. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-ay+b \end{bmatrix}$.

(a)(5 points) Find all a and b such that T is a linear transformation.

$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{b=0}$$

$\boxed{a \text{ is any real number.}}$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & -a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(b)(5 points) Find all a and b such that T is an orthogonal transformation.

T is orthogonal iff the column vectors are orthonormal.

in particular $\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| = 1$

$$\underset{\sqrt{2}}{\overset{\|}{}}$$

contradiction.

Therefore, there are no such a and b.

Problem 10. Let V be the linear space of 3×3 skew-symmetric matrices.

(a)(5 points) Find a basis of V.

$$V = \left\{ \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$$

$$\mathcal{B} = \left(\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{\vec{v}_3} \right)$$

(b)(5 points) Prove that what you found in (a) is a basis of V.

$V = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$: $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are lin independent:

$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = 0 \implies a = b = c = 0$

(c)(5 points) Display a basis of P_k, the set of all polynomials of degree $\leq k$ (k is a non-negative integer).

$$\mathcal{B} = (1, t, \ldots, t^k)$$

(d)(5 points) Find k such that V is isomorphic to P_k.

$$\begin{aligned} \dim P_k &= k+1 \\ \dim V &= 3 \end{aligned} \implies k = 2$$

(e)(5 points) Display an isomorphism between V and P_k.

$$T: V \longrightarrow P_2.$$

$$T\left(\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}\right) = a + bt + ct^2$$

(f)(5 points) Prove that what you found in (d) is an isomorphism.

$$\dim V = \dim P_2 = 3$$

$\ker(T) = 0 \implies T$ is an isomorphism.

Problem 11.(5 points) Let $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4]$ be a matrix in $\mathbb{R}^{5\times 4}$ with linearly independent columns and let $\vec{v}$ be a vector not in $\mathrm{im}(A)$. Show that $\mathrm{rref}(B) = I_5$, where $B = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4\ \vec{v}]$.

$\mathrm{rref}(B) = I_5 \iff$ all column vectors are lin independent.

Since the first four are lin independent, it is enough to show that $\vec{v}$ is not in $\mathrm{span}(\vec{v}_1 \dots \vec{v}_4)$

But $\mathrm{span}(\vec{v}_1 \dots \vec{v}_4) = \mathrm{Im}(A)$ and $\vec{v}$ is not in $\mathrm{Im}(A)$

therefore, indeed $\mathrm{rref}(B) = I_5$.

Problem 12.(10 points) Find the reflection matrix A that transforms $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} -5 \\ 5 \end{bmatrix}$.

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \quad a^2 + b^2 = 1$$

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix} \Rightarrow \begin{cases} 7a + b = -5 \\ 7b - a = 5 \quad \cdot 7 \end{cases} \Rightarrow$$

$$50b = -5 + 35 = 30 \Rightarrow b = \frac{3}{5}$$

$$\left.\begin{array}{l} b = \frac{3}{5} \\ a^2 + b^2 = 1 \end{array}\right\} \Rightarrow a = \pm\frac{4}{5}$$

<u>or</u>

$$a = 7b - 5 = 7 \cdot \frac{3}{5} - 5 = \frac{-4}{5}$$

$$A = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}.$$