Name: SOLUTIONS

Practice Exam on Chapter §4

Math 2270, Fall 2006

Problem 1. Find a basis of the space of all 2×2 matrices A such that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = A \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \iff \begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix} \iff \begin{cases} a = c \\ b = -d \end{cases}$$

The space is: $V = \left\{ \begin{bmatrix} a & b \\ a & -b \end{bmatrix} \middle| \ a, b \in \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right)$

both are lin ind vectors.

Therefore,

$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$ is a basis of V.

Problem 2. (a) Define the kernel of a linear transformation $T \colon V \to W$.

$\ker(T) = \{ f \in V \mid T(f) = 0 \}$.

(b) Define the nullity of a linear transformation $T \colon V \to W$.

If $\dim(\ker(T)) < \infty$, then $\text{nullity}(T) = \dim(\ker(T))$.

(c) Find the kernel and the nullity of the linear transformation $T(f) = f - f'$ from C^∞ to C^∞.

$\ker(T) = \{ f \in C^\infty \mid f = f' \}$

$f = f' \iff f = ae^x$ for some a

$\ker(T) = \{ ae^x \mid a \in \mathbb{R} \} = \text{span}(e^x)$.

$\text{nullity}(T) = \dim(\ker(T)) = 1$, as $\{e^x\}$ is a basis for $\ker(T)$.

Problem 3.
Let P_2 be the space of all polynomials of degree ≤ 2 and consider the following two bases:

$$\mathcal{U} = \{1, t, t^2\} \qquad \text{and} \qquad \mathcal{B} = \{1, t-1, (t-1)^2\}.$$

If $T\colon P_2 \to P_2$ is the linear transformation given by

$$T(f(t)) = f'(t) + f(2t-1),$$

find the following:
(a) The matrix B of T with respect to the basis $\mathcal{B}$.

$$B = \left[[T(1)]_{\mathcal{B}} \quad [T(t-1)]_{\mathcal{B}} \quad [T((t-1)^2)]_{\mathcal{B}} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$T(1) = 0 + 1 = 1$$

$$T(t-1) = 1 + 2t - 1 = 2(t-1)$$

$$T((t-1)^2) = 2(t-1) + (2t-1-1)^2 = 2(t-1) + 4(t-1)^2$$

(b) The change of basis matrix S from the basis $\mathcal{B}$ to the basis $\mathcal{U}$.

$$S = \left[[1]_{\mathcal{U}} \quad [t-1]_{\mathcal{U}} \quad [(t-1)^2]_{\mathcal{U}} \right] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) Is T an isomorphism? Justify your answer.

Yes, because B is an invertible matrix.

(rank B $= 3$).

Problem 4. Prove the following fact: Consider a linear transformation $T\colon V \to V$, and let $\mathcal{B} = \{f_1, \dots, f_n\}$ be a basis of V. Then the $\mathcal{B}$-matrix of transformation B is given by:

$$B = [[T(f_1)]_{\mathcal{B}} \dots [T(f_n)]_{\mathcal{B}}].$$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\mathcal{L}_B"'] & V \arrow[d, "\mathcal{L}_B"] \\ \mathbb{R}^n \arrow[r, "B\cdot"'] & \mathbb{R}^n \end{tikzcd}$$

$B\cdot = \mathcal{L}_B \circ T \circ \mathcal{L}_B^{-1}$

$B = [Be_1 \;\dots\; Be_n]$, where $\vec{e}_1, \dots \vec{e}_n$ is the standard basis of $\mathbb{R}^n$.

$Be_i = \mathcal{L}_B(T(\mathcal{L}_B^{-1}(e_i))) = \mathcal{L}_B(T(f_i)) = [T(f_i)]_{\mathcal{B}}$

for all $i = 1, \dots, n$. ∎

Problem 5. True or False?

(a) There exists a subspace of $\mathbb{R}^{3\times 4}$ isomorphic to P_9.

TRUE! $V = \text{span}\left(\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\right.$

$\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix},$

$\left.\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix}\right)$

(b) If T is a linear transformation from V to V, then the intersection $\text{im}(T)$ and $\ker(T)$ is 0.

False!

If $f \in \text{im}(T) \cap \ker(T) \Rightarrow T(f) = 0$ and $f = T(g)$

$\Rightarrow T(T(g)) = 0$

$\begin{bmatrix}1&1\\-1&-1\end{bmatrix}\begin{bmatrix}1&1\\-1&-1\end{bmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$

$V = \mathbb{R}^{2\times 2}$

$T\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ $\quad T(M) = \begin{bmatrix}1&1\\-1&-1\end{bmatrix} M$

$T^2(M) = \begin{bmatrix}1&1\\-1&-1\end{bmatrix}^2 M = 0$

$\{0\} \neq \operatorname{im}(T) \subseteq \ker(T)$.