Exam # 2 **Fall 2006** **MATH 2270-03**

Instructor: Oana Veliche
Time: 50 minutes

NAME: SOLUTIONS

ID#: ______________________

INSTRUCTIONS
(1) Fill in your name and your student ID number.
(2) Justify all you answers. Correct answers with no justification will not be given any credit.
(3) No books, notes or calculators may be used.

Problem	1	2	3	4	5	6	Total
Max. points	22	17	16	15	10	20	100
Points							

1. Problem. Let P_2 be the space of all polynomials of degree ≤ 2 and consider the following two bases:

$$\mathcal{U} = \{1, t, t^2\} \qquad \text{and} \qquad \mathcal{B} = \{1, t+1, (t+1)^2\}.$$

If $T\colon P_2 \to P_2$ is the linear transformation given by $T(f(t)) = f(t) + f'(t)$, find the following:

(6 points) (a) The matrix B of T with respect to the basis $\mathcal{B}$.

$$B = \left[[T(1)]_{\mathcal{B}} \quad [T(t+1)]_{\mathcal{B}} \quad [T((t+1)^2)]_{\mathcal{B}} \right]$$

$$T(1) = 1 + 0$$

$$T(1+t) = (1+t) + 1$$

$$T((1+t)^2) = (1+t)^2 + 2(1+t)$$

$$\Rightarrow B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

(6 points) (b) The change of basis matrix S from the basis $\mathcal{B}$ to the basis $\mathcal{U}$.

$$S = \left[[1]_{\mathcal{U}} \quad [t+1]_{\mathcal{U}} \quad [(t+1)^2]_{\mathcal{U}} \right]$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

(6 points) (c) Find the nullity and rank of the transformation.

$\ker(T) = \ker(B) = \{0\}. \Rightarrow \text{nullity}(T) = 0$

or $f + f' = 0$

$\Rightarrow f = ce^{-t} \notin P_2$ (if $c \neq 0$) $\Rightarrow f = 0$

$\text{nullity}(T) + \text{rank}(T) = 3$ $\Rightarrow$ $\text{rank}(T) = 3$

(4 points) (d) Is T an isomorphism? Justify your answer.

Yes, because B is invertible.

2. Problem.
Let V be the subspace of $\mathbb{R}^3$ given by $V = \text{span}(\vec{v}_1, \vec{v}_2)$ where

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 25 \\ 0 \\ 25 \end{bmatrix}$$

(6 points) (a) Find an orthonormal basis $\mathcal{U}$ of V.

$$\|\vec{v}_1\| = \sqrt{16+9} = 5 \implies \vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} \frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix} ; \quad \vec{u}_1 \cdot \vec{v}_2 = \begin{bmatrix} \frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix} \begin{bmatrix} 25 \\ 0 \\ 25 \end{bmatrix} = 20 + 15 = 35$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2) \cdot \vec{u}_1 = \begin{bmatrix} 25 \\ 0 \\ 25 \end{bmatrix} - 35 \begin{bmatrix} \frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 25-28 \\ 0 \\ 25-21 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix} ; \quad \|\vec{v}_2^{\perp}\| = 5$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \begin{bmatrix} -\frac{3}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix}$$

(6 points) (b) Write a QR-factorization of the matrix $A = \begin{bmatrix} 4 & 25 \\ 0 & 0 \\ 3 & 25 \end{bmatrix}$.

$$A = QR = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ 0 & 0 \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 5 & 35 \\ 0 & 5 \end{bmatrix}$$

(5 points) (c) Find the matrix of the orthogonal projection onto V.

$$QQ^T = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ 0 & 0 \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Problem. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ and the vector $\vec{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$.

(5 points) (a) Find A^T, the transpose of the matrix A.

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(5 points) (b) Is the matrix $A^T A$ symmetric? What about in general?

$$A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

It is symmetric.

It is also symmetric in general: $(A^TA)^T = A^T (A^T)^T = A^TA$.

(6 points) (c) Find the least-square solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$.

$$\vec{x}^* = (A^TA)^{-1} A^T\vec{b}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 6 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}$$

4. Define the following notions:

(5 points) (a) The *kernel* of a linear transformation $T\colon V \to W$, where V and W are linear spaces.

$\ker(T) = \{ f \in V \mid T(f) = 0 \}.$

(5 points) (b) The *norm* of an element in an inner product space.

$\|f\| = \sqrt{\langle f, f \rangle}$

(5 points) (b') Consider the space of polynomials P with the inner product given by:

$$< f, g >= \int_0^1 f(t)g(t)\ dt.$$

Find the norm of the element $f(t) = t^2$.

$\|t^2\|^2 = \int_0^1 (t^2)^2 dt = \frac{t^5}{5}\Big|_0^1 = \frac{1}{5}$

$\Rightarrow \|t^2\| = \frac{1}{\sqrt{5}}$

(10 points) **5. Prove the following fact.** Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal transformation. If the vectors for all $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are orthogonal, then so are $T(\vec{v})$ and $T(\vec{w})$.

$\|T(\vec{v}) + T(\vec{w})\|^2 = \|T(v+w)\|^2$ T lin transf

$= \|v+w\|^2$ T is orthogonal

$= \|v\|^2 + \|w\|^2$ $\vec{v} \perp \vec{w}$

$= \|T(v)\|^2 + \|T(w)\|^2$ T is orthogonal.

By Pythagorean theorem, it follows that $T(\vec{v}) \perp T(\vec{w})$.

6. True or false ? Justify all your answers.

(5 points) (a) The function $T\colon C^\infty \to C^\infty$ given by $T(f) = 1 + f'$ is a linear transformation.

$T(0) = 1 \neq 0 \Rightarrow$ T is not a linear transf

False

(5 points) (b) For any 3×2 matrices A the following equality holds

$$\dim(\ker(A)) = \dim(\ker(A^T)).$$

$$\left.\begin{array}{l} \dim(\ker(A)) + \operatorname{rank}(A) = 3 \\ \dim(\ker(A^T)) + \operatorname{rank}(A^T) = 2 \\ \operatorname{rank}(A) = \operatorname{rank}(A^T) \end{array}\right\} \Rightarrow \dim(\ker(A)) \neq \dim(\ker(A^T)).$$

False

(5 points) (c) The dimension of the linear space of 2×2 skew-symmetric matrices is 2.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is skew-symmetric $\Leftrightarrow$ $A = -A^T$ $\Leftrightarrow$

$$\begin{cases} a = -a \\ b = -c \\ c = -b \\ d = -d \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = -c \\ d = 0 \end{cases}$$

$\left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \,\middle|\, b \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$ has dimension 1

False

(5 points) (d) If A and B are orthogonal $n \times n$ invertible matrices, then the matrix $B^{-1}A$ is also orthogonal.

$$\left.\begin{array}{l} B \text{ orthogonal} \xRightarrow{\text{Fact}} B^{-1} \text{ orthogonal.} \\ A \text{ orthogonal} \end{array}\right\} \xRightarrow[\text{Fact}]{} B^{-1}A \text{ is orthogonal}$$

True