Practice Final Exam

Math 2270, Fall 2006

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Problem 1. Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $L(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.

(a) Compute the matrix $B = A^T A$.
(b) Find the singular values of the matrix A.
(c) Find an eigenbasis for the matrix B.
(d) Find a singular values decomposition of the matrix A.
(e) Draw the image of the unit circle Ω under the transformation L.
(f) Find the area of $L(\Omega)$.
(g) Is the quadric form $q(\vec{x}) = \vec{x} \cdot B\vec{x}$, defined by the matrix B positive definite ?
(h) Draw the principal axes of q .
(i) Give the formula of the quadric q in the coordinate system defined by the principal axes.
(j) Diagonalize the matrix B.
(k) Using (j) diagonalize the matrix $C = \begin{bmatrix} 0 & 6 \\ 6 & 7 \end{bmatrix}$.

Problem 2. Let $A = \begin{bmatrix} \frac{1}{2} & -k \\ k & \frac{1}{2} \end{bmatrix}$ where k is a real number.

(a) Find all k such that $\vec{0}$ is a stable equilibrium for the dynamical system $\vec{x}(t+1) = A\vec{x}(t)$.
(b) Find the real closed formula for the trajectory $\vec{x}(t+1) = A\vec{x}(t)$ with $k = \frac{\sqrt{3}}{2}$ and $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
(c) Show that the matrices A and $\begin{bmatrix} 1 & -\frac{1}{3} \\ \frac{1}{3} & a \end{bmatrix}$ are not similar for any choices of k and a.

Problem 3. Let A be an $n \times n$ matrix.
(a) Show that A and A^T have the same characteristic polynomial.
(b) Show that if there exists a common eigenbasis for the matrix A and for a matrix B, then $AB = BA$.

Problem 4. Consider the following system $\begin{vmatrix} 3x & + & ky & = & 1 \\ 4x & + & 11y & = & 3 \end{vmatrix}$

(a) Find all k such that the system above has a unique solution.
(b) For $k = 4$, use Cramer's rule to solve the system above.

Problem 5.
Let $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$ be unit vectors in $\mathbb{R}^3$. Find the possible values of $\det[\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$.

Problem 6. Show that if A is an $n \times n$ matrix such that $A^3 + 2A^2 + I_n = 0$, then A is invertible.

Problem 7. Find an orthonormal basis of P_2 with the inner product $< f, g >= \frac{1}{2}\int_{-1}^{1} f(t)g(t)\, dt$.

Problem 8. Show that if A is a square matrix such that $A^T A = AA^T$, then $\ker(A) = \ker(A^T)$.

Problem 9. Show that $T\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^2$ given by $T(A) = \det(A)$ is not a linear transformation.

Problem 10. Let V the linear space of all 3×3 skew-symmetric matrices.
(a) Find a basis of V.
(b) Prove that what you found in (a) is a basis of V.
(c) Find k such that V is isomorphic to P_k.
(d) Display an isomorphism between V and P_k.

Problem 11. Consider $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$ in $\mathbb{R}^5$ and let $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$.
(a) Show that $\text{rank}(A) \neq \text{nullity}(A)$ for any choice of the vectors $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$.
(b) Show that if $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$ are linearly independent, then the columns of the matrix AB are linearly independent for any invertible 3×3 matrix B.

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Problem 1. Consider the following symmetric matrix: $B = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$.

(a) Find the eigenvalues of B.
(b) Find an eigenbasis for the matrix B.
(c) Find an orthonormal eigenbasis for the matrix B.
(d) Diagonalize the matrix B. Check your answer!
(e) Using (d) diagonalize the matrix $C = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$.

Problem 2. Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $L(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.
(a) Find the singular values of the matrix A. Hint: remark that $A^T A = B$, from Problem 1.
(b) Find a singular values decomposition of the matrix A.
(c) Draw the image of the unit circle Ω under the transformation L and find the area of $L(\Omega)$.

Problem 3. (a) Let B be the matrix from Problem 1. Show that the quadric $q(\vec{x}) = \vec{x} \cdot B\vec{x}$ is not indefinite (remark that $B = A^T A$ with A from Problem 2).
(b) Explain why a quadric $q\colon \mathbb{R}^m \to \mathbb{R}$ given by $q(\vec{x}) = \vec{x} \cdot (A^T A)\vec{x}$ is never indefinite for any choice of an $n \times m$ matrix A?

Problem 4. Let $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -k^2 \\ \frac{1}{4} & \frac{\sqrt{3}}{2} \end{bmatrix}$ where k is a real number.
(a) Find all k such that $\vec{0}$ is a stable equilibrium for the dynamical system $\vec{x}(t+1) = A\vec{x}(t)$.

(b) Find the real closed formula for the trajectory $\vec{x}(t+1) = A\vec{x}(t)$ with $k = 1$ and $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 5. Using Cramer's rule solve the following system: $\left| \begin{array}{ccccc} x & + & y & = & 2 \\ x & + & z & = & 0 \\ y & + & z & = & 0 \end{array} \right|$.

Problem 6. Let P_1 be the space of all polynomials of degree ≤ 1 and consider the function:

$$< \, , \, >\colon P_1 \times P_1 \to \mathbb{R}$$

given by the formula: $< f, g >= \frac{1}{2}[f(0)g(0) + f(1)g(1)] + k$, for some k in $\mathbb{R}$.
(a) Show that $< \, , \, >$ is not an inner product if $k \neq 0$.
(b) Find an orthonormal basis of P_1 with the inner product $< f, g >= \frac{1}{2}[f(0)g(0) + f(1)g(1)]$.

Problem 7. Let A be an $n \times n$ matrix with all the entries integers.
(a) Show that if $|\det(A)| = 1$, then A^{-1} has also all the entries integers.
(b) Show that if $A^2 + A + I_n = 0$, then $|\det(A)| = 1$.

Problem 8. Let A be an 3×5 matrix.
(a) Show that if $\vec{v}$ is a vector in $\operatorname{im}(A) \cap \ker(A^T)$, then $\vec{v} = 0$.
(b) Compute $\operatorname{nullity}(A^T)$ if $\operatorname{rank}(A) = 2$.

Problem 9. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x - ay + b \end{bmatrix}$.
(a) Find all a and b such that T is a linear transformation.
(b) Find all a and b such that T is an orthogonal transformation.

Problem 10. Let V be the linear space of 3×3 skew-symmetric matrices.
(a) Find a basis of V.
(b) Prove that what you found in (a) is a basis of V.
(c) Display a basis of P_k, the set of all polynomials of degree $\leq k$ (k is a non-negative integer).
(d) Find k such that V is isomorphic to P_k.
(e) Display an isomorphism between V and P_k.
(f) Prove that what you found in (d) is an isomorphism.

Problem 11. Let $A = [\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3 \; \vec{v}_4]$ be a matrix in $\mathbb{R}^{5\times 4}$ with linearly independent columns and let $\vec{v}$ be a vector not in $\operatorname{im}(A)$. Show that $\operatorname{rref}(B) = I_5$, where $B = [\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3 \; \vec{v}_4 \; \vec{v}]$.

Problem 12. Find the reflection matrix A that transforms $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} -5 \\ 5 \end{bmatrix}$.