Final Exam **Fall 2006** **MATH 2270-03**

Instructor: Oana Veliche
Time: 2 hours

NAME:____________________________________

ID#:_______________________________________

INSTRUCTIONS
(1) Fill in your name and your student ID number.
(2) Justify all your assertions.
(3) No books, notes or calculators may be used.

Problem	1	2	3	4	5	6	7	8	9	10	11	12	Total
Max. points	25	15	25	20	25	20	10	15	15	10	10	15	200
Points													

1. Problem. Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $L(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 3 & 3 \\ -2 & 2 \end{bmatrix}.$$

(a)(10 points) Find the singular values of the matrix A.

(b)(10 points) Find a *singular value decomposition* for A.

(c)(5 points) Describe the image of the unit circle under the transformation L.

2. Prove the following fact: (a) (10 points) Let $L(\vec{x}) = A\vec{x}$ be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Then there exists an orthonormal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ of $\mathbb{R}^m$ such that vectors $L(\vec{v}_1), L(\vec{v}_2), \ldots, L(\vec{v}_m)$ are orthogonal, and have lengths the singular values $\sigma_1, \sigma_2, \ldots \sigma_m$ of matrix A.

(b) (5 points) Find an orthonormal basis as in the fact for the matrix $A = \begin{bmatrix} 3 & 3 \\ -2 & 2 \end{bmatrix}$.

3. Problem. (a) (5 points) Define: *Principal axes of a quadratic form.*

(b)(10 points) Find the principal axes of the quadratic form $q(x_1, x_2) = -3x_1^2 + 6x_1x_2 + 5x_2^2$.

(c)(5 points) Sketch the curve defined by $-3x_1^2 + 6x_1x_2 + 5x_2^2 = 1$.

(d)(5 points) Determine the definiteness of the quadratic form $q(x_1, x_2) = -3x_1^2 + 6x_1x_2 + 5x_2^2$.

4. Problem. Let $U^{2\times 2}$ be the space of all 2×2 upper triangular matrices. Consider the function

$$<\ ,\ >: U^{2\times 2}\times U^{2\times 2}\to \mathbb{R} \quad \text{given by} \quad < A, B >= \operatorname{tr}(A^T B).$$

(a) (5 points) Give an explicit formula of $<\ ,\ >$ in terms of the entries of the matrices.

(b) (10 points) Show that $<\ ,\ >$ is an inner product.

(c) (5 points) Show that $\mathcal{U} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is an orthonormal basis for this inner product.

5. Problem. Consider the following linear subspaces of P_4 (polynomials of degree ≤ 4):

$$V = \{f \text{ in } P_4 | f \text{ is even}\}$$
$$W = \{f \text{ in } P_4 | f \text{ is odd}\}.$$

(a) (10 points) Find a basis of each of the following linear spaces V and W.

(b) (5 points) Are the spaces V and W isomorphic? Justify your answer.

(c) (10 points) If we replace P_4 by P (all polynomials) in the definitions of V and W, are the new spaces V and W isomorphic? Justify your answer.

6. Problem. Let $T\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the linear transformation given by $T(M) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} M$.

(a) (10 points) Find the $\mathcal{U}$-matrix of transformation of T where $\mathcal{U} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is the standard basis of $\mathbb{R}^{2\times 2}$.

(b) (10 points) Show that if $\det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = 0$, then $\det(T) = 0$.

Problem 7. (10 points) Find a reflection matrix A that transforms $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ into $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

Problem 8. Consider the invertible matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) (5 points) Find the matrix $\text{adj}(A)$.

(b) (5 points) Using part (a) find A^{-1}.

(c) (5 points) Check your answer from part (b).

Problem 9. For a 5×6 matrix A find the following:

(a) (5 points) $\text{rank}(A) + \text{nullity}(A)$.

(b) (5 points) $\text{rank}(A) + \text{nullity}(A^T)$.

(c) (5 points) $\text{rank}(A) + \text{nullity}(A^T A)$.

Problem 10. Let A be an 2×3 matrix and let B be an 3×2 matrix such that $AB = I_2$.

(a) (5 points) Show that the columns of B are linearly independent.

(b) (5 points) Show that the columns of A are linearly dependent.

Problem 11. (10 points) Show that if A is a 3×3 matrix with eigenvalues $\lambda_1 = \frac{1}{2}$, $\lambda_2 = -\frac{1}{2}$ and $\lambda_3 = \frac{1}{3}$, then $\lim_{t \to \infty} A^t = \mathbf{0}$.

Problem 12. TRUE or FALSE?

(a) (5 points) If A is a 5×5 skew-symmetric matrix, then $\det(A) = 0$.

(b) (5 points) If A is a 3×2 matrix and $\vec{b}$ is a vector in $\mathbb{R}^3$ such that $A\vec{x} = \vec{b}$ has a unique solution, then for any vector $\vec{c}$ the equation $A\vec{x} = \vec{c}$ has a unique solution.

(c) (5 points) There exist real numbers a and b, such that the matrix $\begin{bmatrix} 1 & -1 \\ 13 & 7 \end{bmatrix}$ is similar to the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.