

Instructor: Oana Veliche

Time: 50 minutes

NAME: _____

ID#: _____

INSTRUCTIONS

- (1) Fill in your name and your student ID number.
- (2) Justify all your answers. Correct answers with no justification will not be given any credit.
- (3) No books, notes or calculators may be used.

Problem	1	2	3	4	5	6	Total
Max. points	22	17	16	15	10	20	100
Points							

1. Problem. Let P_2 be the space of all polynomials of degree ≤ 2 and consider the following two bases:

$$\mathcal{U} = \{1, t, t^2\} \quad \text{and} \quad \mathcal{B} = \{1, t + 1, (t + 1)^2\}.$$

If $T: P_2 \rightarrow P_2$ is the linear transformation given by $T(f(t)) = f(t) + f'(t)$, find the following:

(6 points) (a) The matrix B of T with respect to the basis \mathcal{B} .

(6 points) (b) The change of basis matrix S from the basis \mathcal{B} to the basis \mathcal{U} .

(6 points) (c) Find the nullity and rank of the transformation.

(4 points) (d) Is T an isomorphism? Justify your answer.

2. Problem.

Let V be the subspace of \mathbb{R}^3 given by $V = \text{span}(\vec{v}_1, \vec{v}_2)$ where

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 25 \\ 0 \\ 25 \end{bmatrix}$$

(6 points) (a) Find an orthonormal basis \mathcal{U} of V .

(6 points) (b) Write a QR-factorization of the matrix $A = \begin{bmatrix} 4 & 25 \\ 0 & 0 \\ 3 & 25 \end{bmatrix}$.

(5 points) (c) Find the matrix of the orthogonal projection onto V .

3. Problem. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ and the vector $\vec{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$.

(5 points) (a) Find A^T , the transpose of the matrix A .

(5 points) (b) Is the matrix $A^T A$ symmetric? What about in general?

(6 points) (c) Find the least-square solution \vec{x}^* of the system $A\vec{x} = \vec{b}$.

4. Define the following notions:

(5 points) (a) The *kernel* of a linear transformation $T: V \rightarrow W$, where V and W are linear spaces.

(5 points) (b) The *norm* of an element in an inner product space.

(5 points) (b') Consider the space of polynomials P with the inner product given by:

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt.$$

Find the norm of the element $f(t) = t^2$.

(10 points) **5. Prove the following fact.** Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an orthogonal transformation. If the vectors \vec{v} and \vec{w} in \mathbb{R}^n are orthogonal, then so are $T(\vec{v})$ and $T(\vec{w})$.

6. True or false ? Justify all your answers.

(5 points) (a) The function $T: C^\infty \rightarrow C^\infty$ given by $T(f) = 1 + f'$ is a linear transformation.

(5 points) (b) For any 3×2 matrices A the following equality holds

$$\dim(\ker(A)) = \dim(\ker(A^T)).$$

(5 points) (c) The dimension of the linear space of 2×2 skew-symmetric matrices is 2.

(5 points) (d) If A and B are orthogonal $n \times n$ invertible matrices, then the matrix $B^{-1}A$ is also orthogonal.