Review problems for Final Exam

SOLUTIONS

(1) (a) $y = \dfrac{5x}{3x^4+2}$

$$y' = \frac{5(3x^4+2) - 5x(12x^3)}{(3x^4+2)^2}$$

$$= \frac{15x^4+10-60x^4}{(3x^4+2)^2}$$

$$= \frac{10-45x^4}{(3x^4+2)^2}$$

(b) $y = (x^3+2x^2)^6 e^{3x}$

$$\frac{dy}{dx} = 6(x^3+2x^2)^5(3x^2+4x)\cdot e^{3x} + (x^3+2x^2)^6\cdot e^{3x}$$

$$= (x^3+2x^2)^5\cdot e^{3x}(18x^2+24x+x^3+2x^2)$$

$$= (24x+20x^2+x^3)(x^3+2x^2)^5\cdot e^{3x}$$

(c) $y = \dfrac{3}{\sqrt{x}} - 7\ln(5x^2) + 4e^{8x}$

$$= 3x^{-1/2} - 7\ln 5 - 14\ln x + 4e^{8x}$$

$$\Rightarrow y' = -\frac{3}{2}x^{-3/2} - \frac{14}{x} + 32e^{8x}$$

$$= -\frac{3}{2x\sqrt{x}} - \frac{14}{x} + 32e^{8x}$$

(d) $y = \dfrac{1}{\sqrt{x}}\left(\sqrt[3]{x} - \dfrac{2}{x}\right)$

$$= x^{-1/2}(x^{1/3} - 2x^{-1})$$

$$= x^{-\frac{1}{2}+\frac{1}{3}} - 2x^{-\frac{1}{2}-1}$$

$$= x^{-\frac{1}{6}} - 2x^{-\frac{3}{2}}$$

$$y' = -\frac{1}{6}x^{-\frac{7}{6}} - \cancel{2}\left(-\frac{3}{\cancel{2}}\right)x^{-\frac{5}{2}}$$

$$= -\frac{1}{6x\sqrt[6]{x}} + \frac{3}{x^2\sqrt{x}}$$

(2) $f(x) = \dfrac{2}{x^3} + 5x^4 - 3x^2$

$$= 2x^{-3} + 5x^4 - 3x^2$$

$$f'(x) = -6x^{-4} + 20x^3 - 6x$$

$$f''(x) = 24x^{-5} + 60x^2 - 6$$

$$f'''(x) = -120x^{-6} + 120x$$

$f^{(4)}(x) = 720x^{-7} + 120 = \frac{720}{x^7} + 120$

(3) (a) $\frac{dy}{dx} = 2x + 6y\frac{dy}{dx} + 2 - 3\frac{dy}{dx}$

$(4-6y)\frac{dy}{dx} = 2x+2$

$\frac{dy}{dx} = \frac{2x+2}{4-6y}$

$= \frac{x+1}{2-3y}$

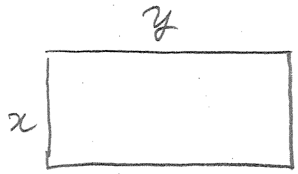
(b) $e^x\frac{dy}{dx} + e^x y + 2x = 4\frac{dy}{dx}$

$\frac{dy}{dx}(4-e^x) = e^x y + 2x$

$\frac{dy}{dx} = \frac{e^x y + 2x}{4-e^x}$

(4) $f'(x) = \frac{(2x^2+1)^3 - x \cdot 3(2x^2+1)^2 \cdot 4x}{(2x^2+1)^6}$

$m = f'(0) = \frac{1}{1} = 1$

(5) 

$P = 2x+2y = 120$

$A = xy$ maximum.

$2x+2y = 120$

$x+y = 60 \Rightarrow y = 60-x$

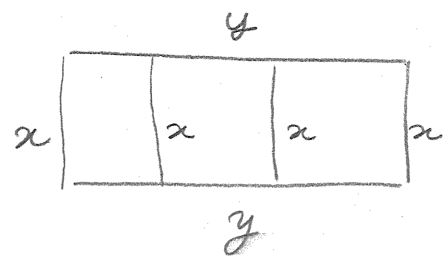
$\Rightarrow A(x) = x(60-x) = 60x - x^2$

$A'(x) = 60-2x$

$A'(x) = 0 \Rightarrow x = 30\ m$, $y = 30\ m$

Check

$A''(x) = -2 < 0 \Rightarrow$ we get a max at $x = 30$.

(6) 

$P = 4x+2y = 180$

$A = xy$

$\Rightarrow y = 90-2x$

$A(x) = x(90-2x)$

$= 90x - 2x^2$

$A'(x) = 90-4x$

$A'(x) = 0 \Rightarrow x = \frac{90}{4} = \frac{45}{2}$

$= 22.5$ ft.

$y = 90-45 = 45$ ft.

$A''(x) = -4 < 0 \Rightarrow$ we get max at $x = 22.5$.

(7) $f(x) = \dfrac{2x^2-3x+4}{(4x+3)(x-1)^2}$

Vertical asymptotes: $4x+3=0$

$\Rightarrow$ $x = -\frac{3}{4}$

$x-1=0 \Longrightarrow x=1$.

Horizontal asymptotes:

degree of numerator $=2$
degree of denominator $=3$

$2<3 \Rightarrow$

$y=0$ is an horizontal asymptote.

(8) $f(x) = 2x^4-6x^2$

$f'(x) = 8x^3-12x$

$= 4x(2x^2-3)$

$f'(x)=0 \Rightarrow x=0,\ x=\pm\sqrt{\frac{3}{2}}$

x		$-\sqrt{\frac{3}{2}}$		0		$\sqrt{\frac{3}{2}}$	
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\searrow$	$-\frac{9}{2}$	$\nearrow$	0	$\searrow$	$-\frac{9}{2}$	$\nearrow$

$f(-\sqrt{\frac{3}{2}}) = 2\cdot\left(\frac{3}{2}\right)^2 - \cancel{6}^{3}\cdot\frac{3}{\cancel{2}}$

$= \cancel{2}\cdot\frac{9}{\cancel{4}_2} - 9 = -\frac{9}{2}$

$f(\sqrt{\frac{3}{2}}) = -\frac{9}{2}$

Relative minimum points:

$\left(-\sqrt{\frac{3}{2}}, -\frac{9}{2}\right)$ and $\left(\sqrt{\frac{3}{2}}, \frac{9}{2}\right)$

relative maximum point: $(0,0)$.

(9) $f(x) = -x^3+3x^2-2$

$f'(x) = -3x^2+6x$

$f''(x) = -6x+6$

$f''(x)=0 \Rightarrow x=1$

$f(1) = -1+3-2=0$

x		1	
$f''(x)$	$+$	0	$-$
$f(x)$	$\cup$	0	$\cap$

$\Rightarrow$ $(1,0)$ is an inflection point.

(10) $f(x) = \frac{1}{3}x^3-9x+2$

$f'(x) = x^2-9$

$f'(x)=0 \Rightarrow x=\pm 3$

Only 3 is the inside of $[0,4]$.

$f(0)=2$

$f(3) = \frac{1}{\cancel{3}}\cdot \cancel{27}^{9} - 27+2$

$= -18+2 = -16$

$f(4) = \frac{1}{3}\cdot 4^3 - \frac{3}{}9\cdot 4 + 2$

$= \frac{4(16-18)}{3} + 2$

$= -\frac{8}{3} + \frac{3}{2} = -\frac{8}{3} + \frac{6}{3}$

$= -\frac{2}{3}.$

abs. maximum value : 2

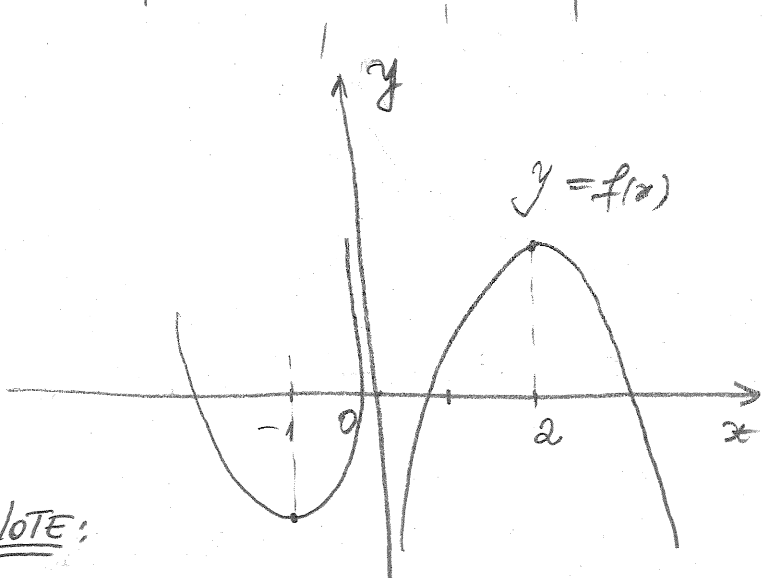
$\Rightarrow$ abs max. point : $(0,2)$

abs minimum value $= -16$

$\Rightarrow$ abs min point : $(3,-16)$.

(11)

x		-1		0		2	
$f'(x)$	$-$	0	$+$	/	$+$	0	$-$
$f''(x)$	$+$		$+$	/	$-$		$-$
$f(x)$	↘		↗		⌒		↘



NOTE:

There are many correct answers.

(12) $A = Pe^{rt}$

$A = 20{,}000\cdot e^{0.04\cdot 10}$

$= 20{,}000\, e^{0.4}$

$= \$29836.49395$

(13) (a) $\int x^2\left(4x - \frac{3}{\sqrt{x}} + 2\right)dx$

$= \int 4x^3 - 3x^{3/2} + 2x^2\,dx$

$= x^4 - 3\frac{x^{5/2}}{5/2} + \frac{2x^3}{3} + C$

$= x^4 - \frac{6}{5}x^2\sqrt{x} + \frac{2}{3}x^3 + C$

(b) $\int \frac{6}{2y-1}dy = 3\int \frac{2}{2y-1}dy$

$= 3\ln|2y-1| + C$

(c) $\int 3x^2(3x^3+4)^3 dx$

$= \frac{1}{3}\int 9x^2(3x^3+4)^3 dx$

$= \frac{1}{3}\,\frac{(3x^3+4)^4}{4} + C$

$= \frac{1}{12}(3x^3+4)^4 + C$

(d) $\displaystyle\int_1^2 \frac{1}{(2x-1)^3}\,dx =$

$u = 2x-1$ $\quad x=1 \Rightarrow u=1$

$du = 2\,dx$ $\quad x=2 \Rightarrow u=3$

$$= \int_1^3 u^{-3}\cdot\frac{du}{2} = \frac{1}{2}\,\frac{u^{-2}}{-2}\Big|_1^3$$

$$= -\frac{1}{4u^2}\Big|_1^3 = -\frac{1}{36} + \frac{\cancel{1}^{9}}{4} = \frac{8}{36}$$

$$= \frac{2}{9}$$

(e) $\displaystyle\int \frac{1}{x\ln(2x)}\,dx$

$u = \ln(2x)$

$du = \frac{2}{2x}dx = \frac{1}{x}dx$

$$= \int \frac{1}{u}\cdot du = \ln|u| + C$$

$$= \boxed{\ln|\ln(2x)| + C}$$

(f) $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$

$$= \int \frac{1}{(x+1)(x+2)}\,dx = \ldots$$

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$1 = A(x+2) + B(x+1)$

$x=-1 \Rightarrow A=1$

$x=-2 \Rightarrow B=-1$

$$\Rightarrow \quad \ldots = \int \frac{1}{x+1} - \frac{1}{x+2}\,dx$$

$$= \ln|x+1| - \ln|x+2| + C$$

$$= \boxed{\ln\left|\frac{x+1}{x+2}\right| + C}$$

(14) (a) $\displaystyle\int_3^\infty \frac{2}{x^6}\,dx = \lim_{b\to\infty}\int_3^b 2x^{-6}\,dx$

$$= 2\lim_{b\to\infty} \frac{x^{-5}}{-5}\Big|_3^b$$

$$= 2\lim_{b\to\infty}\left(-\frac{1}{5b^5} + \frac{1}{5\cdot 3^5}\right)$$

$$= \frac{2}{5\cdot 27\cdot 9} = \frac{2}{1215}$$

$$\begin{array}{r} 27\cdot \\ 45 \\ \hline 135 \\ 108 \\ \hline 1215 \end{array}$$

(b) $\displaystyle\int_2^3 \frac{1}{\sqrt{x-2}}\,dx$

$$= \lim_{a\to 2^+}\int_a^3 (x-2)^{-1/2}\,dx$$

$$= \lim_{a\to 2^+} 2(x-2)^{1/2}\Big|_a^3$$

$$= \lim_{a\to 2^+} 2\cdot 1 - 2\sqrt{a-2} = 2$$

(c) $\displaystyle\int_{-\infty}^{\infty} x^2 e^{-x^3}\,dx = \int_{-\infty}^{0} x^2 e^{-x^3}\,dx + \int_0^\infty x^2 e^{-x^3}\,dx$

$$\int_{-\infty}^{0} x^2 e^{-x^3}\,dx = -\frac{1}{3}\lim_{a\to-\infty}\int_{a}^{0} -3x^2 e^{-x^3}\,dx$$

$$= -\frac{1}{3}\lim_{a\to-\infty} e^{-x^3}\Big|_{a}^{0}$$

$$= -\frac{1}{3}\lim_{a\to-\infty}\left(1 - e^{-a^3}\right) = \infty$$

$\Rightarrow$ The integral is divergent.

(15) Area $= \int_{0}^{\ln 2} x e^{-x}\,dx =$

$u = x \Rightarrow du = dx$

$dv = e^{-x}dx \Rightarrow v = -e^{-x}$

$$= -xe^{-x}\Big|_{0}^{\ln 2} - \int_{0}^{\ln 2} -e^{-x}\,dx$$

$$= -\ln 2\, e^{-\ln 2} - e^{-x}\Big|_{0}^{\ln 2}$$

$$= -\frac{\ln 2}{2} - \frac{1}{2} + 1 = \frac{1}{2} - \frac{\ln 2}{2}$$

units^2

(16) x = # of units
p = price

x	p
6000	\$325
8000	\$300

$p = mx + b$, $m = \frac{325-300}{6000-8000} = \frac{25}{-2000}$

$= -0.0125$

$325 = \frac{25}{-2000}\cdot \overset{3}{\cancel{6000}} + b \qquad b = 325 + 75$
$= 400$

$\Rightarrow$ The demand function is:

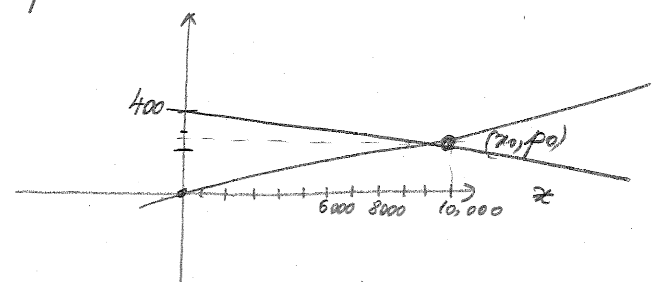
$p = -0.0125x + 400$

We find next the equilibrium point:

$0.0275x = -0.0125x + 400$

$0.04x = 400 \Rightarrow x_0 = \frac{\overset{100}{\cancel{400}}}{\underset{\frac{1}{100}}{\cancel{0.04}}} = 10,000$ units.

$p_0 = 0.0275 \cdot 10,000 = \$275.$



Consumer surplus $= \int_{0}^{10,000} -0.0125x + 400 - 275\,dx$

$$= -0.0125\frac{x^2}{2} + 125x\Big|_{0}^{10,000}$$

$= 625,000 + 1875\,000 = 2,500,000$

Producer surplus $= \int_{0}^{10000} 275 - 0.0275x\,dx$

$$= 275x - 0.0275\frac{x^2}{2}\Big|_{0}^{10,000}$$

$= 1,375,000.$

(17) $f(x) = \int x^2 \ln 2x \, dx = \dots$

$u = \ln 2x \Rightarrow du = \frac{1}{x} dx$

$dv = x^2 dx \Rightarrow v = \frac{x^3}{3}$

$\dots = \frac{x^3}{3} \ln(2x) - \int \frac{x^2}{3} dx$

$= \frac{x^3}{3} \ln(2x) - \frac{1}{9} x^3 + C$

$f(1) = \frac{\ln 2}{3} - \frac{1}{9} + c = 2$

$\Rightarrow c = 2 + \frac{1}{9} - \frac{\ln 2}{3}$

$= \frac{19}{9} - \frac{\ln 2}{3}$

$\Rightarrow \boxed{f(x) = \frac{x^3}{3} \ln(2x) - \frac{1}{9} x^3 + \frac{19}{9} - \frac{\ln 2}{3}}$

(18) Amount of annuity

$= e^{0.04 \cdot 25} \int_0^{25} 1000 \, e^{-0.04 \cdot t} dt$

$= e \cdot 1000 \cdot \frac{e^{-0.04t}}{-0.04} \Big|_0^{25}$

$= \frac{e \cdot 1000}{-0.04} (e^{-1} - 1)$

$= (e-1) \cdot 25000 \cong$

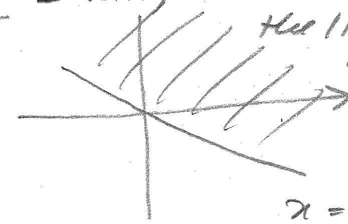
(19) Area $= \int_0^\infty e^{-3x} dx = \lim_{b \to \infty} \int_0^b e^{-3x} dx$

$= \lim_{b \to \infty} \frac{e^{-3x}}{-3} \Big|_0^b$

$= \lim_{b \to \infty} \left(-\frac{1}{3e^{3b}} + \frac{1}{3} \right) = \frac{1}{3}$ units^2

(20) Domain = semiplane without the line $x = -y$.

$x + y > 0$



$x = -y$

Range $[0, \infty)$

(21) (a) $\frac{\partial f}{\partial x} = 4yz + 3x^2y^2z + 3x^2$

(b) $f_{xz} = 4y + 3x^2y^2$

(c) $f_{xyz} = f_{xzy} = 4 + 6yx^2$

(d) $\frac{\partial f}{\partial y}\Big|_{(1,2,0)} = (4xz + 2x^3yz)\Big|_{(1,2,0)}$

$= 0$

(22) $R(x) = 44x - 0.02x^2$

(a) $R'(x) = 44 - 0.04x$

$R'(x) = 0 \iff x = \frac{44}{0.04}$

$= 1100$ units.

$R''(x) = -0.04 < 0 \Rightarrow$ we get the maximum

(b) $R(1100) = 44 \cdot 1100 -$

$- 0.02 \cdot (1100)^2 = 1100 \, (44 - 22)$

$= 22 \cdot 1100 = \$24200$

(c)

x		1100	
$R'(x)$	$+$	0	$-$
$R(x)$	$\nearrow$	24200	$\searrow$

$R(1000)$ will be maximum

$R(1000) = 44 \cdot 1000 - 0.02 \cdot (1000)^2$

$= 1000(44-20) = \$24{,}000$.