Review problems for Exam #3
Solutions

(1) (a) $\int x(x^3-2)\,dx$

$= \int x^4 - 2x\,dx$

$= \dfrac{x^5}{5} - x^2 + C$

(b) $\int \dfrac{x^2}{x^3+1}\,dx = \dots$

$u = x^3+1$

$du = 3x^2\,dx$

$\Rightarrow x^2\,dx = \dfrac{du}{3}$.

$\dots = \int \dfrac{1}{u}\cdot\dfrac{du}{3} = \dfrac{1}{3}\ln|u| + C$

$= \dfrac{1}{3}\ln|x^3+1| + C$

OR

$\int \dfrac{x^2}{x^3+1}\,dx = \dfrac{1}{3}\int \dfrac{3x^2}{x^3+1}\,dx$

$= \dfrac{1}{3}\ln|x^3+1| + C$

(c) $\int 5x^3 e^{x^4-1}\,dx$

$= \dfrac{5}{4}\int 4x^3 e^{x^4-1}\,dx$

$= \dfrac{5}{4} e^{x^4-1} + C$ OR

$\int 5x^3 e^{x^4-1}\,dx =$

$u = x^4-1$

$du = 4x^3\,dx$

$\Rightarrow x^3\,dx = \dfrac{du}{4}$

$= \int 5e^u \cdot \dfrac{du}{4} = \dfrac{5}{4}\int e^u\,du$

$= \dfrac{5}{4}e^u + C$

$= \dfrac{5}{4}e^{x^4-1} + C$

(d) $\int 5\sqrt{t} - \dfrac{3}{t} + \dfrac{4}{t^3}\,dt$

$= \int 5t^{1/2} - \dfrac{3}{t} + 4t^{-3}\,dt$

$= \dfrac{5t^{3/2}}{3/2} - 3\ln|t| + 4\dfrac{t^{-2}}{-2} + C$

$= \dfrac{10}{3}t\sqrt{t} - 3\ln|t| - \dfrac{2}{t^2} + C$

(e) $\int \dfrac{4}{e^{2x}}\,dx = \int 4e^{-2x}\,dx$

$= 4\dfrac{e^{-2x}}{-2} + C = -\dfrac{2}{e^{2x}} + C$

(f) $\displaystyle\int_2^{10} \dfrac{3}{\sqrt{5x-1}}\,dx =$

$u = 5x-1$

$du = 5\,dx$

$x = 2 \Rightarrow u = 3$

$x = 10 \Rightarrow u = 7$

$$= \int_3^7 \frac{3}{\sqrt{u}} \cdot \frac{du}{5} = \frac{3}{5}\int_3^7 u^{1/2}\, du$$

$$= \frac{\cancel{3}}{5} \frac{u^{3/2}}{\frac{\cancel{3}}{2}} \Big|_3^7 = \frac{2}{5}\left(7\sqrt{7} - 3\sqrt{3}\right)$$

(g) $\displaystyle\int_0^1 (2x^2-3)^4 x\, dx = \ldots$

$u = 2x^2 - 3$

$du = 4x\, dx$

$x = 0 \Rightarrow u = -3$

$x = 1 \Rightarrow u = 2 - 3 = -1$

$$\ldots = \int_{-3}^{-1} u^4 \cdot \frac{du}{4} =$$

$$= \frac{1}{4}\frac{u^5}{5}\Big|_{-3}^{-1} = \frac{1}{20}\left(-1 + 3^5\right)$$

$$= \frac{728}{20} = \frac{364}{10} = 36.4$$

(h) $\displaystyle\int \frac{v^2+v}{\underbrace{1-3v^2-2v^3}_{u}} dv$ $\quad u \Rightarrow u' = -6v - 6x^2$

$$= \frac{1}{-6}\int \frac{-6(v^2+v)}{1-3v^2-2v^3} dv$$

$$= -\frac{1}{6} \ln|1-3v^2-2v^3| + C$$

(i) $\displaystyle\int_1^2 \left(1+\frac{1}{w}\right)^2 \cdot \frac{1}{w^2} dw = \ldots$

$u = 1 + \frac{1}{w} \Rightarrow du = -\frac{1}{w^2} dw$

$w = 1 \Rightarrow u = 2$

$w = 2 \Rightarrow u = \frac{3}{2}$

$$\ldots = \int_2^{3/2} u^2(-du) = -\frac{u^3}{3}\Big|_2^{3/2}$$

$$= -\frac{27}{8 \cdot 3} + \frac{8/8}{3} = \frac{64-27}{24}$$

$$= \frac{37}{24}$$

(j) $\displaystyle\int \frac{1}{x(\ln x)^2} dx = \ldots$

$u = \ln x$

$du = \frac{1}{x} dx$

$$\ldots = \int \frac{1}{u^2} du = \frac{u^{-1}}{-1} + C$$

$$= -\frac{1}{\ln x} + C$$

(k) $\displaystyle\int_0^{\ln 2} \frac{e^x}{1+e^x} dx$

$$= \ln(1+e^x)\Big|_0^{\ln 2} =$$

$= \ln(1+e^{\ln 2}) - \ln(1+e^{0})$

$= \ln(1+2) - \ln(2)$

$= \ln\frac{3}{2}$

(e) $\int_{-1}^{3} |x-2|\,dx$

$= \int_{-1}^{2} -(x-2)\,dx + \int_{2}^{3} x-2\,dx$

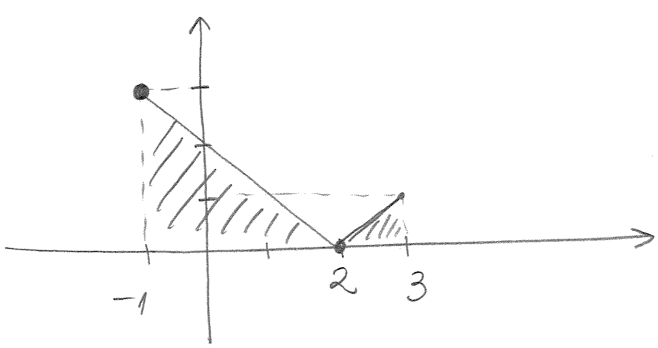
$= \left(-\frac{x^2}{2}+2x\right)\Big|_{-1}^{2} + \left(\frac{x^2}{2}-2x\right)\Big|_{2}^{3}$

$= -\cancel{\frac{4}{2}} + \cancel{4} + \frac{1}{2} + \cancel{2} + \frac{9}{2} \cancel{-6}$

$-\cancel{\frac{4}{2}} + \cancel{4}$

$= 5$

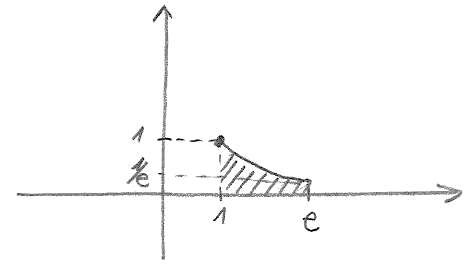
OR



A = the shaded area

$= \frac{3\cdot 3}{2} + \frac{1\cdot 1}{2} = \frac{9}{2} + \frac{1}{2} = 5$

(2) (a)



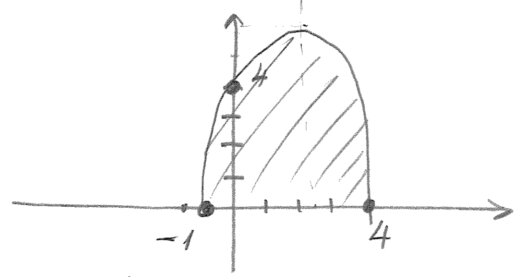
$A = \int_{1}^{e} \frac{1}{x}\,dx = \ln|x|\Big|_{1}^{e} = \ln e - \ln 1$

$= 1 \text{ units}^2$

(b) $y = 4+3x-x^2$

$y=0 \Rightarrow (4-x)(1+x)=0$

$\Rightarrow x=4,\ x=-1$



$A = \int_{-1}^{4} 4+3x-x^2\,dx =$

$= 4x + \frac{3x^2}{2} - \frac{x^3}{3}\Big|_{-1}^{4}$ $\quad \frac{16\cdot 4}{64}$

$= 16 + \frac{3\cdot \overset{8}{\cancel{16}}}{\cancel{2}} - \frac{4^3}{3} + 4 - \frac{3}{2} - \frac{1}{3}$

$= \overset{6}{44} - \overset{3}{\frac{3}{2}} - \overset{2}{\frac{65}{3}} = \frac{264-9-130}{6}$

$= \frac{\overset{41}{\cancel{123}}}{\underset{2}{\cancel{6}}} = 20.5 \text{ units}^2$

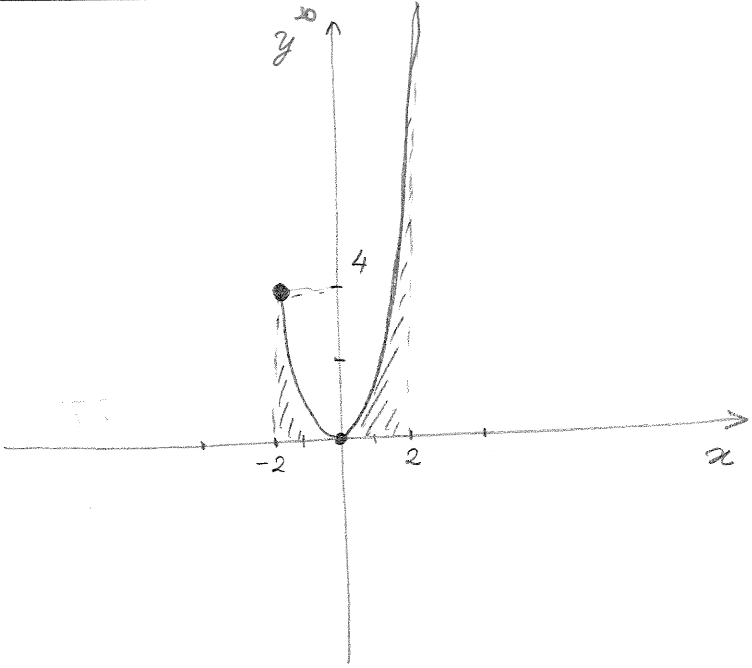
(c) $y = x^3+3x^2 \quad x=-2,\ x=2$

$\Updownarrow$

$y = x^2(x+3) \quad \left[y=0 \Rightarrow \begin{array}{l} x=0 \\ x=-3 \end{array}\right]$

$y(-2) = -8+12 = 4$

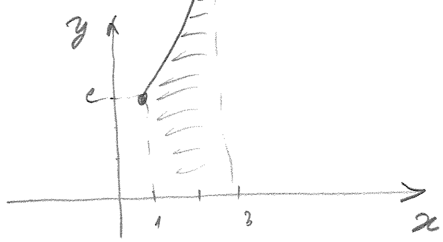
$y(2) = 8+12 = 20$



$$A=\int_{-2}^{2} x^3+3x^2\,dx = \frac{x^4}{4}+x^3\Big|_{-2}^{2}$$

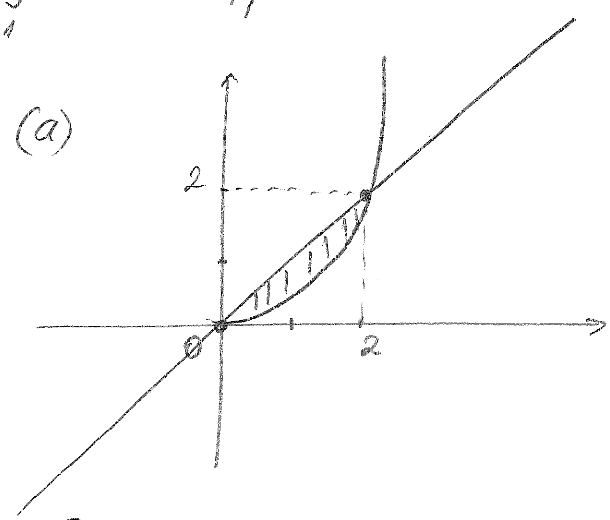
$$= \frac{2^4}{4}+8-\left(\frac{2^4}{4}-8\right) = 16 \text{ units}^2$$

(d)



$$\int_1^3 e^x\,dx = e^x\Big|_1^3 = e^3-e^1 = e^3-e \text{ units}^2$$

(3) (a)



$$x^2=2x \iff$$

$$x(x-2)=0 \iff$$

$$x=0 \text{ or } x=2$$

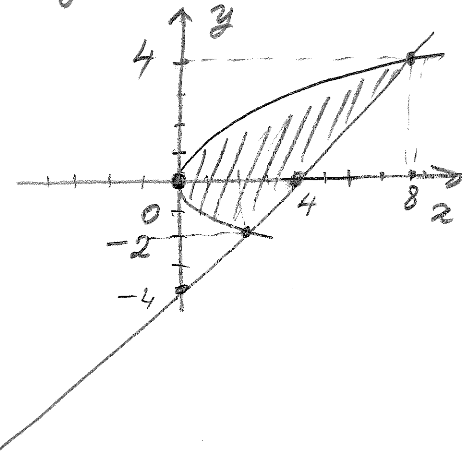
$$A=\int_0^2 2x-x^2\,dx = x^2-\frac{x^3}{3}\Big|_0^2$$

$$= 4-\frac{8}{3} = \frac{12-8}{3} = \frac{4}{3} \text{ units}^2$$

(b) $y=x-4$ and $y^2=2x$

$$x=y+4$$

$$x=\frac{y^2}{2}$$



We get now the intersection points:

$$y+4=\frac{y^2}{2} \iff y^2=2y+8$$

$$y^2-2y-8=0$$

$$(y-4)(y+2)=0$$

$$y=4 \text{ and } y=-2$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$(x=8 \qquad\qquad x=2)$$

$$A=\int_{-2}^{4} y+4-\frac{y^2}{2}\,dy$$

$$= \frac{y^2}{2}+4y-\frac{y^3}{6}\Big|_{-2}^{4}$$

$$= \frac{16}{2}+16-\frac{16\cdot\cancel{4}^{\,2}}{\cancel{6}_{\,3}}-\frac{4}{2}+8-\frac{\cancel{8}^{\,4}}{\cancel{6}_{\,3}}$$

$$= (8+16-2+8)-\frac{32+4}{3}$$

$$= 30-12 = 18 \text{ units}^2$$

(c) $x+5y=6$, $y=\frac{1}{x}$

$$\Downarrow$$

$y=\frac{6-x}{5}$. We get the intersection points:

$$\frac{1}{x} = \frac{6-x}{5} \Rightarrow$$

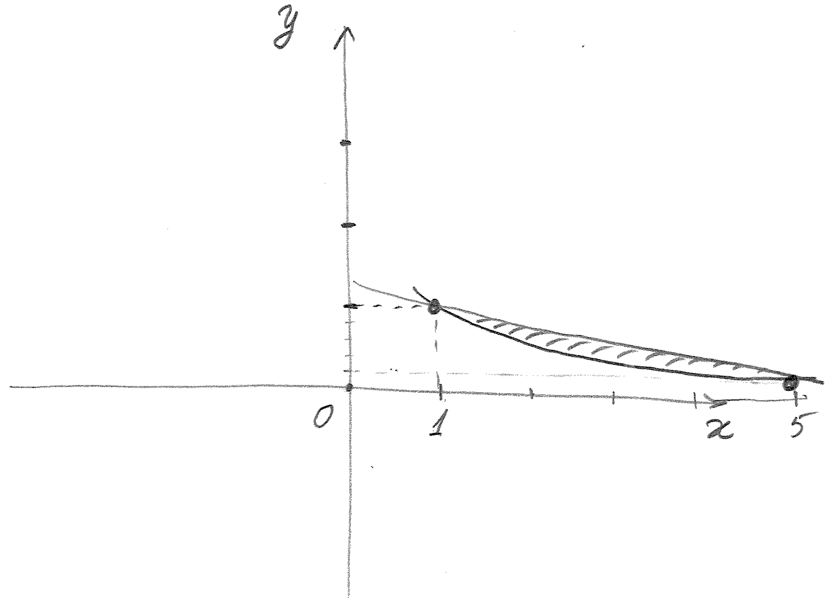
$$5 = 6x - x^2 \Rightarrow x^2 - 6x + 5 = 0$$

$$(x-1)(x-5) = 0$$

$x = 1$ OR $x = 5$

$$x = 1 \Rightarrow y = \frac{1}{1} = 1$$

$$x = 5 \Rightarrow y = \frac{1}{5}$$



$$A = \int_1^5 \frac{6-x}{5} - \frac{1}{x}\, dx =$$

$$= \int_1^5 \frac{6}{5} - \frac{x}{5} - \frac{1}{x}\, dx = \frac{6}{5}x - \frac{x^2}{10} - \ln x \Big|_1^5$$

$$= 6 - \frac{25}{10} - \ln 5 - \frac{6}{5} + \frac{1}{10} + \cancel{\ln 1}^{\,0}$$

$$= \frac{24}{5} - \frac{24}{10} - \ln 5 = \boxed{2.4 - \ln 5 \text{ units}^2}$$

(d) $y = \sqrt{x}$ and $y = x^2$

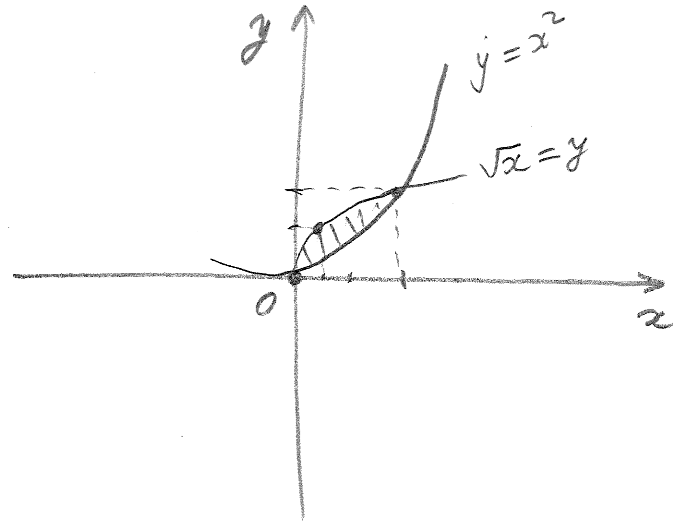
We get the intersection points:

$\sqrt{x} = x^2 \qquad x \geq 0.$

$x = x^4 \Rightarrow x = 0$ or $x^3 = 1$ i.e $x = 1$.

$$x = 0 \Rightarrow y = 0$$

$$x = 1 \Rightarrow y = 1$$



$$x = \frac{1}{4} \Rightarrow y = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\Rightarrow y = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

$$A = \int_0^1 \sqrt{x} - x^2\, dx$$

$$= \frac{x^{3/2}}{3/2} - \frac{x^3}{3} \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3} \text{ units}^2}$$

(4) $$f' = \frac{x^2 + 2x + 5}{x - 1}$$

$$f = \int \frac{x^2 + 2x + 5}{x - 1}\, dx = \int x + 3 + \frac{8}{x-1}\, dx$$

$$\begin{array}{r|l} & x + 3 \\ \hline x - 1 & x^2 + 2x + 5 \\ & -(x^2 - x) \\ \hline & 3x \\ & -(3x - 3) \\ \hline & 8 \end{array}$$

$= \frac{x^2}{2} + 3x + 8 \ln|x-1| + C$

$f(2) = 2 \implies$

$\frac{\cancel{4}}{\cancel{2}}^{2} + 6 + \underbrace{8 \ln 1}_{0} + C = 2$

$C = \cancel{2} - \cancel{2} - 6 \implies C = -6$

$\implies f(x) = \frac{x^2}{2} + 3x + 8 \ln|x-1| - 6$

(5) Amount of an annuity $=$

$e^{rT} \int_0^T c(t) e^{-rt} \, dt$

Amount of an annuity $=$

$= e^{0.09 \cdot 20} \int_0^{20} 500 e^{-0.09t} \, dt$

$= e^{1.8} \cdot \frac{500 e^{-0.09t}}{-0.09} \Big|_0^{20}$

$= \frac{\cancel{e^{1.8}} \cdot 500 \cdot \cancel{e^{-1.8}}}{-0.09} + \frac{e^{1.8} \cdot 500}{0.09}$

$= \frac{500}{0.09} (e^{1.8} - 1)$

$\cong \$28{,}053.59$

(6) $f(-x) = (-x)^2 + |-x|$

$= x^2 + |x| = f(x) \implies$

$f(x)$ is even.

$\int_{-2}^{2} x^2 + |x| \, dx = 2 \int_0^2 x^2 + |x| \, dx$

$= 2 \int_0^2 x^2 + x \, dx = 2 \cdot \left(\frac{x^3}{3} + \frac{x^2}{2} \right) \Big|_0^2$

$= 2 \cdot \left(\frac{8}{3} + \frac{\cancel{4}^{\,3}}{\cancel{2}} \right) = 2 \cdot \frac{8+6}{3} = \frac{28}{3}$

(7) $\int_{100}^{104} \frac{200-x}{50} \, dx = \int_{100}^{104} 4 - \frac{x}{50} \, dx$

$= 4x - \frac{1}{50} \cdot \frac{x^2}{2} \Big|_{100}^{104}$

$= 4(104-100) - \frac{1}{100}(104^2 - 100^2)$

$= 4 \cdot 4 - \frac{1}{100}(104-100)(104+100)$

$= 8 - \frac{4 \cdot 204}{100} = 8 - 8{,}16$

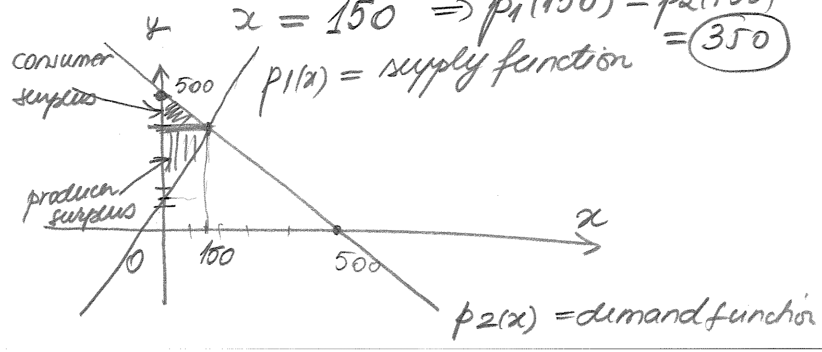
$= -0.16 \$$

(8) $p_1(x) = p_2(x) \implies$

$1.25x + 162.5 = 500 - x$

$2.25x = 500 - 162.5$

$x = 150 \implies p_1(150) = p_2(150) = 350$



Consumer surplus $= \int_0^{150} (\text{demand fct} - \text{price})\,dx$

$$= \int_0^{150} 500 - x - 350\,dx = 150x - \frac{x^2}{2}\Big|_0^{150}$$

$$= (150)^2 - \frac{(150)^2}{2} = \frac{(150)^2}{2} =$$

$$= \frac{22500}{2} = 11{,}250.$$

11250

Producer surplus $= \int_0^{150} (\text{price} - \text{supply fct})\,dx$

$$= \int_0^{150} 350 - 1.25x - 162.5\,dx$$

$$= 187.5x - 1{,}25\frac{x^2}{2}\Big|_0^{150}$$

$$= 187.5 \cdot 150 - 1.25 \cdot \frac{(150)^2}{2}$$

$$= 28125 - 14062.5 = 14{,}062.5.$$

(9) (a) $\displaystyle\int_0^1 \frac{x}{(x+4)^4}\,dx =$

$\boxed{u = x+4 \Rightarrow x = u-4 \quad du = dx}$

$\boxed{x = 0 \Rightarrow u = 4 \quad x = 1 \Rightarrow u = 5}$

$$= \int_4^5 \frac{u-4}{u^4}\,du = \int_4^5 u^{-3} - 4u^{-4}\,du$$

$$= -\frac{u^{-2}}{-2} - 4\frac{u^{-3}}{-3}\Big|_4^5$$

$$= \frac{1}{2u^2} + \frac{1}{12u^3}\Big|_4^5$$

$$= \frac{\frac{5\cdot 64}{1}}{2\cdot 25} + \frac{\frac{64}{1}}{12\cdot 125} - \frac{\frac{4\cdot 125}{1}}{2\cdot 16} - \frac{\frac{125}{1}}{12\cdot 64}$$

$$= \frac{320 + 64 - 500 - 125}{96000}$$

$$= -\frac{241}{96000}$$

(b) $\displaystyle\int_0^1 x\sqrt[3]{1-x}\,dx = \ldots$

$\boxed{u = 1-x \Rightarrow x = 1-u \quad du = -dx}$

$\boxed{x = 0 \Rightarrow u = 1 \quad x = 1 \Rightarrow u = 0}$

$$\ldots = \int_1^0 (1-u)\,u^{1/3}(-du)$$

$$= \int_1^0 u^{4/3} - u^{1/3}\,du = \frac{u^{7/3}}{7/3} - \frac{u^{4/3}}{4/3}\Big|_1^0$$

$$= -\frac{3}{7} + \frac{3}{4} = \frac{21-12}{28} = \frac{9}{28}$$

(c) $\displaystyle\int x e^{4x}\,dx = x\cdot\frac{e^{4x}}{4} - \int \frac{e^{4x}}{4}\,dx$

$\boxed{u = x \Rightarrow du = dx \quad dv = e^{4x}dx \Rightarrow v = \frac{e^{4x}}{4}}$

$$= \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C$$

(d) $\int \frac{x}{e^x}dx = \int x e^{-x} dx =$

$$u = x \Rightarrow du = dx$$
$$dv = e^{-x}dx \Rightarrow v = -e^{-x}$$

$$= -xe^{-x} - \int -e^{-x}dx$$

$$= -xe^{-x} - e^{-x} + C$$

(e) $\int x^4 \ln x \, dx =$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$
$$dv = x^4 dx \Rightarrow v = \frac{x^5}{5}$$

$$= \frac{1}{5}x^5 \ln x - \int \frac{1}{x} \cdot \frac{x^5}{5} dx$$

$$= \frac{1}{5}x^5 \ln x - \frac{x^5}{25} + C$$

(f) $\int_1^e x(\ln x)^2 dx =$

$$u = (\ln x)^2 \Rightarrow du = 2\ln x \cdot \frac{1}{x}dx$$
$$dv = x\,dx \Rightarrow v = \frac{x^2}{2}$$

$$= \frac{x^2}{2}(\ln x)^2\Big|_1^e - \int_1^e \cancel{2}\ln x \, \frac{1}{x} \cdot \frac{x^2}{\cancel{2}} dx$$

$$= \frac{e^2}{2} - \int_1^e x \ln x \, dx =$$

Apply integration by parts again.

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$
$$dv = x\,dx \Rightarrow v = \frac{x^2}{2}$$

$$= \frac{e^2}{2} - \left[\frac{x^2}{2}\ln x\Big|_1^e - \int_1^e \frac{1}{x} \cdot \frac{x^2}{2} dx\right]$$

$$= \cancel{\frac{e^2}{2}} - \cancel{\frac{e^2}{2}} + \int_1^e \frac{x}{2} dx$$

$$= \frac{x^2}{4}\Big|_1^e = \frac{e^2}{4} - \frac{1}{4} = \frac{e^2 - 1}{4}$$

(g) $\int x\sqrt{x+2}\, dx =$

$$u = x+2 \Rightarrow x = u-2$$

$$= \int (u-2)u^{1/2} du$$

$$= \int u^{3/2} - 2u^{1/2} du$$

$$= \frac{u^{5/2}}{5/2} - 2\frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C$$

$$= \frac{2}{5}(x+2)^2\sqrt{x+2} - \frac{4}{3}(x+2)\sqrt{x+2} + C$$

(10) (a) Actual income =

$$\int_0^4 150{,}000 + 75{,}000t \, dt$$

$$= 150{,}000t\Big|_0^4 + 75{,}000\frac{t^2}{2}\Big|_0^4$$

$$= 600{,}000 + 75{,}000 \cdot 8$$

$$= \$1{,}200{,}000.$$

(9)

(b) Present value =

$$= \int_0^4 \underbrace{(150{,}000+75{,}000t)}_{u}\,\underbrace{e^{-0.04t}\,dt}_{dv}$$

$$\Downarrow$$

$$du = 75{,}000\,dt \qquad v = \frac{e^{-0.04t}}{-0.04}$$

$$= (150{,}000+75{,}000t)\cdot\frac{e^{-0.04t}}{-0.04}\Big|_0^4$$

$$- \underbrace{\int_0^4 75{,}000\cdot\frac{e^{-0.04t}}{-0.04}\Big|_0^4}$$

$$\frac{75{,}000}{(0.04)^2}\cdot e^{-0.04t}\Big|_0^4$$

$$= \frac{300{,}000\cdot e^{-0.16}}{-0.04} - \frac{150{,}000}{-0.04} -$$

$$- \frac{75{,}000}{(0.04)^2}e^{-0.16} + \frac{75{,}000}{(0.04)^2}$$

$$= \$1{,}094{,}142.26$$

(11) (a) $\dfrac{2}{1-x^2} = \dfrac{2}{(1-x)(1+x)}$

$$= \frac{A}{1-x} + \frac{B}{1+x}$$

$$2 = A(1+x) + B(1-x)$$

$2 = A+B$ $\quad\Rightarrow A = B = 1$

$0 = A-B$

$$\int\frac{2}{1-x^2}dx = \int\frac{1}{1-x}dx + \int\frac{1}{1+x}dx$$

$$= \ln|1-x| + \ln|1+x| + C$$

$$= \ln|1-x^2| + C$$

(b) $\dfrac{1}{3x^2+x} = \dfrac{1}{x(3x+1)}$

$$= \frac{A}{x} + \frac{B}{3x+1}$$

$$1 = A(3x+1) + Bx$$

$x=0 \Rightarrow A=1$

$x=-\frac{1}{3} \Rightarrow B=-3$

$$\Rightarrow \int\frac{1}{3x^2+x}dx = \int\frac{1}{x} - \frac{3}{3x+1}dx$$

$$= \ln|x| - \ln|3x+1| + C$$

$$= \ln\left|\frac{x}{3x+1}\right| + C$$

(c) $\dfrac{x+1}{(x-2)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2}$

$$x+1 = A(x-2) + B$$

$$= Ax - 2A + B$$

$$\Rightarrow A = 1,\quad B = 2A+1 = 3 \Longrightarrow$$

$$\int\frac{x+1}{(x-2)^2}dx = \int\frac{1}{x-2} + \frac{3}{(x-2)^2}dx$$

$$= \ln|x-2| - \frac{3}{x-1} + C$$

(d) $$\frac{x^2-x}{x^2+x+1} =$$

$$= \frac{x^2+x+1 \quad -2x-1}{x^2+x+1}$$

$$= 1 - \frac{2x+1}{x^2+x+1}$$

$$\int \frac{x^2-x}{x^2+x+1}\,dx = \int 1 - \frac{2x+1}{x^2+x+1}\,dx$$

$$= x - \ln(x^2+x+1) + C \,.$$