Moduli of products of curves

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The main result of this article is the construction of several connected, irreducible components of the moduli space of stable surfaces. These components parameterize products of stable curves and are constructed from the corresponding moduli spaces of stable curves. The essential results are that products of stable curves are stable surfaces, and all infinitesimal deformations of a product of stable varieties (of any dimension) come from the deformations of the factors. In particular, this gives a fairly simple proof that the moduli space of minimal surfaces of general type with fixed Hilbert polynomial may have arbitrarily many connected components.

All schemes are defined over the field **C**. A *variety* is a connected, reduced, and separated scheme of finite type, not necessarily assumed irreducible. A *family* is a flat morphism of varieties. The base space of a miniversal deformation will be called the *Kuranishi space*, following the convenient terminology from analytic geometry.

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1 Semi-log-canonical singularities and moduli of stable surfaces

First a preliminary remark: if X is a S_2 variety which is Gorenstein in codimension 1, then the extension of the dualizing sheaf of the Gorenstein locus (which is locally free) is a reflexive sheaf on X which corresponds to a Weil divisor K_X. X is called **Q**-Gorenstein if some multiple of K_X is Cartier. Using these definitions, one may define the class of singularities to be studied.

Definition 1.1. A variety X is said to have *semi-log canonical* (slc) singularities if

1. X is **Q**-Gorenstein;
2. X is S_2;
3. X has at worst normal crossing singularities is codimension 1;
4. there exists a good desingularization (i.e. a desingularization whose exceptional fibers are simple normal crossings divisors) $f : Y \to X$ such that in the formula

$$K_Y \equiv f^* K_X + \sum a_i E_i$$

all of the a_i are positive.

The definition of the moduli space of stable surfaces is somewhat technical. In fact, there are at least two different definitions which make sense but may lead to nonisomorphic moduli spaces. The original definition

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(as well as the definition of slc singularities) appeared in [KSB88]. Later the conditions that a family of stable varieties should satisfy were amended in [Kol90].

In the case of Gorenstein varieties, these subtleties do not occur, and the two moduli functors do not differ. However, it is only in special cases that the moduli space of minimal surfaces of general type can be compactified by adding stable surfaces with only Gorenstein singularities to the moduli problem. This suffices for the purposes of this paper, so only this special case is considered.

Definition 1.2. The moduli functor of stable Gorenstein surfaces is a functor from schemes to sets which assigns to a scheme B the set of isomorphism classes of flat, proper morphisms $X \to B$ whose fibers are Gorenstein schemes with slc singularities and whose relative dualizing sheaf $\omega_{X/B}$ is ample. An isomorphism of families X, X' is an B-isomorphism ϕ together with an isomorphism of $\omega_{X/B}$ with $\phi^*\omega_{X'/B}$.

This article considers a smaller functor. Let M_{g_1,g_2} be the functor which assigns to B the set of isomorphism classes of flat proper morphisms $X \to B$ whose fibers are products of stable curves of genera g_1 and g_2. The results of this article will show that this functor is coarsely representable by a connected and projective variety and that it is an open and closed subfunctor of the moduli functor of stable Gorenstein surfaces.

2 Deformations of products

In this section, some general deformation-theoretic results are proved about products of varieties. These results are formal and primarily homological. The goal is to show that under some conditions, the small deformations of a product of varieties are obtained by deforming the factors. The condition we use is the following:

Definition 2.1. A *stable variety* of index N is a projective (pure) n-dimensional variety X with slc singularities and such that $\omega_X^{[N]}$ is an ample line bundle. In particular, X is reduced, connected, S_2, and $\mathbf{Q}$-Gorenstein.

Let $X = Y_1 \times Y_2$ be a variety which is the product of two stable varieties Y_1 and Y_2; let π_i denote the projection map to Y_i. This notation will be fixed throughout this section.

The following "rigidity lemma" will be useful:

Lemma 2.2. *If $h : X_1 \times X_2 \to B_1 \times B_2$ is a surjective morphism of products of stable varieties, then after possibly renumbering, h can be written as the product of maps $h_i : X_i \to B_i$, $i = 1, 2$.*

Proof. This follows from the fact that the tangent space to the scheme $\mathrm{Hom}(X_i, B_j)$ at the equivalence class $[f]$ of a morphism is $H^0(X_i, f^*\mathcal{T}_{B_j})$ which vanishes due to the stability assumption (a stable variety has no infinitesimal automorphisms). A morphism h as in the hypothesis which is not a product would be a non-trivial deformation of some morphism $f : X_i \to B_j$. □

In particular, it follows that, up to renumbering, a product of stable curves can be written as a product of curves in a unique way. This depends on the stability assumption, as there exist abelian surfaces which can be written in distinct ways as the product of elliptic curves.

Recall that the space of first order deformations of a variety X is given by $T^1(X) = \mathbf{Ext}^1(\mathbf{L}_X, \mathcal{O}_X)$, where $\mathbf{L}_X$ is the cotangent complex of X ([Ill71]). Stable curves and products thereof have local complete intersection singularities only, and $\mathbf{L}_X$ reduces to the cotangent sheaf Ω_X. However, using the formalism of the cotangent complex and total derived functors streamlines and generalizes the proof.

Theorem 2.3. *Every first-order deformation of X is the product of a first order deformation of Y_1 with a first order deformation of Y_2 if Y_1 and Y_2 are stable varieties.*

Proof. Let $\mathbf{L}_X$, $\mathbf{L}_{Y_1}$, and $\mathbf{L}_{Y_2}$ denote the cotangent complexes of X, Y_1, and Y_2, respectively. Denote by **Ext** the hyperext groups. We need to show:

$$\mathbf{Ext}^1_X(\mathbf{L}_X, \mathcal{O}_X) \cong \mathbf{Ext}^1_{Y_1}(\mathbf{L}_{Y_1}, \mathcal{O}_{Y_1}) \oplus \mathbf{Ext}^1_{Y_2}(\mathbf{L}_{Y_2}, \mathcal{O}_{Y_2})$$

by [Ill71], III.1.2.0.

By *ibid.*, II.2.2.3,

$$\begin{aligned} \mathbf{Ext}^1(\mathbf{L}_X, \mathcal{O}_X) &\cong \mathbf{Ext}^1(\pi_1^*\mathbf{L}_{Y_1} \oplus \pi_2^*\mathbf{L}_{Y_2}, \mathcal{O}_X) && (1)\\ &\cong \mathbf{Ext}^1(\pi_1^*\mathbf{L}_{Y_1}, \pi_1^*\mathcal{O}_{Y_1}) \oplus \mathbf{Ext}^1(\pi_2^*\mathbf{L}_{Y_2}, \pi_2^*\mathcal{O}_{Y_2}), && (2) \end{aligned}$$

The following computation finishes the proof:

$$\begin{aligned} \mathbf{Ext}^1(\pi_1^*\mathbf{L}_{Y_1}, \pi_1^*\mathcal{O}_{Y_1}) &\cong H^1[\mathrm{RHom}(\pi_1^*\mathbf{L}_{Y_1}, \pi_1^*\mathcal{O}_{Y_1})] && (3)\\ &\cong H^1[R\Gamma(X, R\mathcal{H}om(\pi_1^*\mathbf{L}_{Y_1}, \pi_1^*\mathcal{O}_{Y_1}))] && (4)\\ &\cong H^1[R\Gamma(X, \pi_1^* R\mathcal{H}om(\mathbf{L}_{Y_1}, \mathcal{O}_{Y_1}))] && (5)\\ &\cong H^1[R\Gamma(Y_2, R\pi_{2*}\pi_1^* R\mathcal{H}om(\mathbf{L}_{Y_1}, \mathcal{O}_{Y_1}))] && (6)\\ &\cong H^1[R\Gamma(Y_2, \mathcal{O}_{Y_2} \otimes R\Gamma(Y_1, R\mathcal{H}om(\mathbf{L}_{Y_1}, \mathcal{O}_{Y_1})))] && (7)\\ &\cong H^1[R\Gamma(Y_2, \mathcal{O}_{Y_2}) \otimes \mathrm{RHom}(\mathbf{L}_{Y_1}, \mathcal{O}_{Y_1})] && (8)\\ &\cong [H^0(R\Gamma(Y_2, \mathcal{O}_{Y_2})) \otimes H^1(\mathrm{RHom}(\mathbf{L}_{Y_1}, \mathcal{O}_{Y_1})] && (9)\\ &\quad \oplus [H^1(R\Gamma(Y_2, \mathcal{O}_{Y_2})) \otimes H^0(\mathrm{RHom}(\mathbf{L}_{Y_1}, \mathcal{O}_{Y_1})]\\ &\cong \mathbf{Ext}^1(\mathbf{L}_{Y_1}, \mathcal{O}_{Y_1}) \oplus [H^1(Y_2, \mathcal{O}_{Y_2}) \otimes \mathrm{Der}(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_1})] && (10)\\ &\cong \mathbf{Ext}^1(\mathbf{L}_{Y_1}, \mathcal{O}_{Y_1}) && (11) \end{aligned}$$

The steps are justified as follows: the composition of derived functors rule ([Har66], II.5.3) justifies steps (4), (6), and part of (8). Step (5) follows from the flatness of π_1 using *ibid.*, II.5.8. Step (7) is *ibid.*, II.5.12. Step (8) follows from *ibid.*, II.5.16. Step (9) is the Künneth formula. Step (10) follows from properness of Y_2 and [Ill71], II.1.2.4.3. Step (11) follows from the fact that stable varieties have no infinitesimal automorphisms, so $\mathrm{Der}(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_1})$ vanishes. ☐

Note that the stability hypothesis is essential: suppose the Y_i were both smooth elliptic curves. Then one may replace all of the $\mathrm{Ext}^1(\Omega, \mathcal{O})$ on X and the Y_i with $H^1(\mathcal{T})$. The tangent sheaf of an abelian variety is trivial, so $h^1(Y_1, \mathcal{T}_{Y_1}) + h^1(Y_2, \mathcal{T}_{Y_2}) = h^1(Y_1, \mathcal{O}_{Y_1}) + h^1(Y_2, \mathcal{O}_{Y_2}) = 2$, but by Hodge theory, $h^1(X, \mathcal{T}_X) = 2h^1(X, \mathcal{O}_X) = \dim_{\mathbf{C}} H^1(X, \mathbf{C}) = 4$.

Corollary 2.4. *The Kuranishi space of a product of finitely many stable curves is smooth.*

Proof. This follows from the fact that the deformations of stable curves are unobstructed, and from the above result shows that the only infinitesimal deformations of the product come from the factors, and are consequently unobstructed. ☐

3 Products of stable varieties

This section concerns the products of stable varieties. These products are also stable, and we can obtain more information in the case of products of curves.

Proposition 3.1. *The product of stable curves is a stable surface. More specifically, the product of stable curves has only normal crossings and degenerate cusps as singular points.*

Proof. The question is analytically local. The singularities of stable curves are nodes. Since the product of a smooth point on a curve with a node is a normal crossing singularity, it suffices to check that the product of a node with itself is slc.

One must compute a semiresolution of the scheme Spec $\mathbf{C}[x, y, w, z]/(xy, wz)$. This scheme is the affine cone over a cycle of rational curves, so blowing up the cone point is a semiresolution with exceptional locus being a cycle of rational curves. Therefore the singularity at the cone point is a degenerate cusp. All of the other singular points are plainly normal crossings.

Having checked that the singularities are slc, the stability assertion is simply the ampleness of the canonical bundle, which follows from the ampleness of the canonical bundles of the factors. □

This is a special case of the following more general result, but the proof with coordinate rings is retained to see exactly what singularities occur in the case of products of stable curves.

Theorem 3.2. *Let Y_1 and Y_2 be stable varieties. Then $Y_1 \times Y_2$ is a smoothable stable variety.*

Proof. The ampleness of the canonical class of $Y_1 \times Y_2$ is immediate. It remains to verify that products of slc singularities are slc. First, the conditions of $\mathbf{Q}$-Gorenstein, S_2 and normal crossings in codimension 1 are clearly preserved under taking products. Let $f : X \to Y_1$ be a desingularization. Then write

$$K_X = f^* K_{Y_1} + \sum a_i E_i$$

where the E_i are exceptional. The a_i are all greater than or equal to -1 since Y_1 is slc. Therefore the exceptional divisors of the product morphism $X \times Y_2 \to Y_1 \times Y_2$ occur with coefficient greater than or equal to -1. Since X is smooth and Y_2 is slc, $X \times Y_2$ is slc. Therefore the discrepancies of a resolution of $X \times Y_2$ are all greater than or equal to -1, so $Y_1 \times Y_2$ is slc, since a resolution of $X \times Y_2$ is also a resolution of $Y_1 \times Y_2$. □

4 Main results

The main theorems below are stated and proved in the case of the product of two curves for ease of notation. However, the proofs generalize to the product of finitely many curves. Denote by M_g the moduli functor of stable curves of genus g. This functor is known to be coarsely representable by a projective variety.

Theorem 4.1. *Let $g_1, g_2 \geq 2$. If $g_1 \neq g_2$, then M_{g_1,g_2} is isomorphic to $M_{g_1} \times M_{g_2}$.*

Proof. Taking fibered products gives a natural transformation $M_{g_1} \times M_{g_2} \to M_{g_1,g_2}$. This natural transformation is relatively representable. By 2.3, it is étale. By 2.2 it is injective on geometric points, that is $M_{g_1}(k) \times M_{g_2}(k) = M_{g_1,g_2}(k)$ when k is an algebraically closed field. The natural transformation is proper since $M_{g_1} \times M_{g_2}$ is proper. It follows that the functors are isomorphic (a proper, étale, injective morphism is an isomorphism) and consequently that M_{g_1,g_2} is coarsely representable. □

The results of the previous sections show that M_{g_1,g_2} is a subfunctor of the moduli space of stable Gorenstein surfaces.

A similar argument proves:

Theorem 4.2. *$M_{g,g}$ is isomorphic to the symmetric square of the functor M_g if $g \geq 2$ and hence coarsely representable by the symmetric square of the corresponding moduli space of curves.*

Corollary 4.3. *Given $m > 0$, there exists a pair of Chern numbers such that the moduli space of stable Gorenstein surfaces with these Chern numbers has at least m connected components.*

Proof. Given $m > 0$, there exists a positive integer N which factors in at least m distinct ways as a product of two distinct factors. Choose m pairs (a_i, b_i) such that $(a_i - 1)(b_i - 1) = N$. Let C_{a_i} and C_{b_i} be smooth curves of genus a_i and b_i, respectively for each i.

The products

$$\begin{array}{c} C_{a_1} \times C_{b_1} \\ \vdots \\ C_{a_m} \times C_{b_m} \end{array}$$

have the same Chern numbers, since these invariants may be computed from the genera of the curves. However, these curves belong to different components of the moduli space since the genera chosen are distinct. □

One could also draw several easy corollaries of the theorem from the deep results in [HM98] concerning the moduli spaces of stable curves; in particular:

Corollary 4.4. *M_{g_1,g_2} is of general type if g_1 and g_2 are distinct and both greater than 23.*

Also, the rational Picard group is not as simple as in the case of curves.

Corollary 4.5. *Let g_1 and g_2 be distinct integers greater than 2. Then* $\text{Pic } M_{g_1,g_2} \otimes \mathbf{Q} \cong (\text{Pic } M_{g_1} \otimes \mathbf{Q}) \times (\text{Pic } M_{g_2} \otimes \mathbf{Q})$.

Proof. The moduli spaces of curves are integral schemes of finite type. Furthermore, $H^1(M_g, \mathbf{C}) = 0$ (see, e.g. [AC98]). Their singularities are at worst finite quotient singularities, since the Kuranishi spaces for curves are smooth and stable curves have a finite automorphism group. Since finite quotient singularities are DuBois, the results of [DB81] imply that $H^1(M_g, \mathbf{C}) \to H^1(M_g, \mathcal{O}_{M_g})$ is surjective, so the latter group is zero. The result that the Picard group of the product decomposes as the product of Picard groups under these hypotheses is [Har77] ex. III.12.6. □

Specifically, for the moduli spaces of curves, the rational Picard group is freely generated by the Hodge class and the classes of the components of the boundary divisor [AC87]. The rational Picard group of the product has too high a rank for the same to be true.

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