1. Find the Fourier sine series of the function

$$f(x) = \begin{cases} 0, & x = 0, \pi \\ 100 & 0 < x < \pi \end{cases}$$

extended to be periodic of period 2π.

2. Suppose $u(x,t)$ gives the temperature at a point x units from the left endpoint of a wire of length π at time t. Then u satisfies the *heat equation*: $u_t(x,t) = c^2 u_{xx}(x,t)$ (c is some constant based on the wire). Suppose now that the ends of the wire are stuck in ice: then the boundary conditions $u(0,t) = u(\pi,t) = 0$ are satisfied (at least until the ice melts!). Suppose further that there is an initial temperature distribution in the wire $u(x,0) = f(x)$.

a. Suppose $u(x,t) = X(x)T(t)$. Explain how to obtain the equations:

$$\begin{aligned} X'' - kX &= 0 \\ T' - kc^2 T &= 0 \end{aligned}$$

for some constant k.

b. Suppose the wire is not a constant temperature of 0 degrees (i.e., $f(x) \neq 0$). Explain why k must be negative, so the equations may be rewritten:

$$\begin{aligned} X'' + \mu^2 X &= 0 \\ T' + \mu^2 c^2 T &= 0 \end{aligned}$$

where $\mu = -\sqrt{k}$.

c. Show that $X(x) = a \sin \mu x$, and that μ must be an integer. Set $X_n(x) = \sin nx$.

d. Solve $T_n'(t) + (nc)^2 T_n(t) = 0$ to find $T_n(t)$. (Answer: $T_n(t) = b_n e^{-n^2 c^2 t}$ for some constant b_n.

e. Write a general formula for u, and use the initial condition to show that

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx$$

f. Suppose the wire is heated uniformly to 100 degrees, then the ends are stuck in ice. Assume $c = 1$ for the wire. Find $u(x,t)$.

3. Show that $u(x,t) = -\kappa u_x(x,t) + k(x,t)$ (where κ is a given constant and $k(x,t)$ is a given function) is solved by $u(x,t) = e^{-rt} f(x - \kappa t)$, where f is any smooth function.

4. Solve the equation $x^2 u_x + y^2 u_y = 0$ (your solution will involve an arbitrary function f).

5. Solve the homogeneous wave equation for a string with length 1 and $c = 1$ with initial conditions $u(x,0) = \sin \pi x$, $u_t(x,0) = 0$.