Ryan Barrett

Professor Michael van Opstall

History of Math

8 July 2008

An Exploration into the Golden Ratio

The golden ratio, golden section, golden mean, golden number, divine proportion, divine section, extreme and mean ratio, are all names given to the ratio which today we denote as φ (phi). Mark Barr, who lived and worked in the 20th century, was the first individual to denote the golden ratio by φ. It is said that he initially used this notation because it is the first letter of Greek sculptor Phidias' name.

The golden ratio has interested mathematicians, geometers, artists, and architects, alike, for at least 2,400 years. Ancient Greek Mathematicians first studied φ because it appeared often in commonly constructed geometrical designs. Regular pentagrams and pentagons employ the golden ratio, and thus the Greeks frequently attributed the discovery of the ratio to Pythagoras or his followers ("the Pythagoreans"), given their symbol, the pentagram.

The golden mean appears in its written form for the first time in Euclid's *Elements*: "A straight line is said to have been cut 'in extreme mean and ratio' when, as the whole line is to the greater segment, so is the greater to the less." The term "extreme mean and ratio" was hence the principally used way to describe the ratio from the 3rd century BCE until about the 18th century of the Common Era. Luca Pacioli's *Divina Poportione*, written in 1509, essentially began what we today think of as the "golden ratio." His work captured the mysteriousness of the number and all the intrigue with which it is associated.

To determine the value of the ratio, one needs simply to employ algebraic techniques and a substitution provided by the ratio, itself. $\frac{a}{b} = \varphi$ yields $a = \beta\varphi$, the substitution quantity.

Making the aforementioned substitution, $a = \beta\varphi$,

$$\frac{a+\beta}{\alpha} = \frac{\alpha}{\beta} \rightarrow \frac{b\varphi+\beta}{\beta\varphi} = \frac{\beta\varphi}{\beta}.$$

Canceling the constant b gives,

$$\frac{\varphi+1}{j} = j \ .$$

Multiplying φ over produces,

$$\varphi^2 = j \ + 1.$$

And finally, moving $\varphi + 1$ to the left side, one obtains,

$$\varphi^2 - j \ - 1 = 0.$$

Employing the quadratic formula to this equation gives as its only positive solution,

$\varphi = \frac{1+\sqrt{5}}{2}$, the golden ratio.

φ is irrational. A myriad of proofs exist to exhibit this irrationality, however, only one will be shown here, as it is compact and clean. This proof is a derivation of the irrationality of $\sqrt{5}$. Suppose it is already known that the square root of a non-square natural number is irrational. This was essentially known in ancient Greece when the side and diagonal of a square were shown to be incommensurable. The proof, itself, is one by contradiction, and relies on the closure of rational numbers under addition and multiplication. That is, given $a,b \in /$, $a+\beta \in /$ and $ab \in /$. Assume $\varphi = \frac{1+\sqrt{5}}{2}$ is rational. So then, by this closure of ¤ , must $2\left(\frac{1+\sqrt{5}}{2} - \frac{1}{2}\right) = \sqrt{5}$ be rational, which is an obvious contradiction. Thus φ must be irrational, as originally proposed.

The golden ratio is inextricably linked to Leonardo of Pisa's (Fibonacci's) sequence of numbers known as the Fibonacci numbers. Fibonacci (1170 – 1250) was an Italian mathematician, considered by some "the most talented mathematician of the Middle Ages." His sequence first appears in the *Liber Abaci*, Fibonacci's *Book of Calculations*. A problem involving the growth of hypothetical rabbits under idealized conditions is posed and solved in the *Liber Abaci*.

The problem proceeds as follows: Suppose there exists one pair of rabbits – a female and a male – in a field. This pair can mate and produce one other pair of rabbits at the end of the month. At the end of the next month, each pair of rabbits can produce another single pair of fertile rabbits. The process continues indefinitely, giving the following string of numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, …

The following recurrence relation, F_n, defines the preceding sequence where:

$$F_n = \begin{cases} 0 & \iota\phi\ \nu = 0; \\ 1 & \iota\phi\ \nu = 1; \\ \Phi_{\nu-1} + \Phi_{\nu-2} & \iota\phi\ \nu > 1. \end{cases}$$

The question remains, then, how are the golden ratio and the Fibonacci sequence related? The answer lies in the construction of the Fibonacci sequence, itself. To start, one must take the limit of the sequence of Fibonacci numbers. That is, $\lim\limits_{n\to\infty}\frac{\Phi_{v+1}}{\Phi_v}$. To assume this quantity exists would be presumptuous from a theoretical standpoint; however, its proof is beyond the scope of this paper. Therefore, theory aside, let $\lim\limits_{n\to\infty}\frac{\Phi_{v+1}}{\Phi_v}=\xi$, where x is some variable not equal to zero. By construction, then, given $F_n=\Phi_{v-1}+\Phi_{v-2}$, the former relation can be manipulated as follows:

$$x=\lambda\mu_{v\to\infty}\frac{\Phi_{v+1}}{\Phi_v}=\lambda\mu_{v\to\infty}\frac{\Phi_v+\Phi_{v-1}}{\Phi_v}=\lambda\mu_{v\to\infty}\frac{\Phi_v}{\Phi_v}+\lambda\mu_{v\to\infty}\frac{\Phi_{v-1}}{\Phi_v}=\lambda\mu_{v\to\infty}1+\lambda\mu_{v\to\infty}\frac{\Phi_{v-1}}{\Phi_v}\,.$$

Then by our definition of x, and the fact that $\lim\limits_{n\to\infty}1=1$, we are left with the equation:

$$x=1+\frac{1}{\xi}\to x-1-\frac{1}{\xi}=0\,\cdot$$

Multiplying x through (which we know we can do because it is non-zero) yields,

$$x^2-\xi-1=0$$

which has for its positive solution the golden ratio, φ. Thus $x=\lambda\mu_{v\to\infty}\frac{\Phi_{v+1}}{\Phi_v}=\varphi$.

There exists, also, as for all sequences defined by linear recurrence, a closed form of the Fibonacci sequence which has become known as Binet's formula:

$$F_n=\frac{\varphi^v-(1-\varphi)^v}{\sqrt{5}}=\frac{\varphi^v-(-\varphi)^{-v}}{\sqrt{5}}$$, where φ is the known constant derived from the golden ratio. This formula, although credited to Binet, was known almost a century earlier by the famous mathematician Abraham de Moivre.

Now that the positive root of the quadratic equation has been studied, it becomes necessary to delve into the realm of negativity; namely, the "conjugate" root of the quadratic equation for φ. As mentioned before, $\varphi^2 - \varphi - 1 = 0$ has two roots. One of which is the golden ratio, the other of which is $\frac{1-\sqrt{5}}{2} = 1 - \varphi \approx -0.6180339887$. The absolute value of this quantity corresponds to the length ratio taken in reverse order. It is commonly referred to as the golden ratio conjugate. It is denoted by the symbol Φ (Phi), and possesses, like the golden ratio, itself, some interesting properties.

$$\Phi = \frac{1}{\varphi} = \varphi - 1 \approx 0.6180339887.$$

This illustrates the property of the golden ratio among positive numbers. Namely,

$$\frac{1}{\varphi} = \varphi - 1$$

and its inverse,

$$\Phi = \left(\frac{1}{\Phi}\right) - 1 \rightarrow \frac{1}{\Phi} = 1 + \Phi .$$

Following the discussion on the nature of the golden ratio, it seems in order to talk about the golden rectangle, which is, in itself, an application of the golden ratio. The figure below represents a golden rectangle. By cutting the rectangle in such a way that a square is produced on one side, one creates another golden rectangle within the first's perimeter. Repeating this process indefinitely produces infinitesimally small golden rectangles. Successive points dividing a golden rectangle into squares lie on a logarithmic spiral. This spiral, in keeping with the tradition, is sometimes called the golden spiral.

Johannes Kepler, a German mathematician and astronomer studied what is today called the Kepler triangle. This is a right triangle with edge lengths in geometric progression; that is, the ratio of the sides of the triangle is $1 : \sqrt{\varphi} : \varphi$, where φ represents the golden ratio. Kepler triangles combine two key concepts in mathematics: the Pythagorean theorem and the golden ratio. Both the Pythagorean theorem and the golden ratio fascinated Kepler. He once said that, "Geometry has two great treasures: one is the theorem of Pythagoras, the other the division of a line into mean and extreme ratio. The first we may compare to a mass of gold, the second we may call a precious jewel."

The construction of the Kepler triangle is derived directly from the equation, which defines the golden ratio; namely, $\varphi^2 = j + 1$. Manipulating this equation, one arrives at: $(\varphi)^2 = (\sqrt{j})^2 + (1)^2$. Thus, using the theorem of Pythagoras, it becomes apparent that one can produce a right triangle with legs of length $\sqrt{\varphi}$ and 1 and hypotenuse φ. An astounding example of a Kepler triangle can be extracted from the Great Pyramid of Giza. The medial right triangle composed of the apothem, semi-base and height of the Great Pyramid closely approximates a Kepler triangle. Whether this anomaly was by design or merely happened upon is the subject of controversy.

The golden ratio transcends pure mathematics into many other areas of study including art, architecture, nature, music, and aesthetics. Leonardo da Vinci's illustrations in *De Divina Proportione* (*On the Divine Proportion*) seem, to many scholars, to represent his use of the golden proportion in his paintings. Some suggest his *Mona Lisa* employs such proportions in its geometric equivalents. In nature, angles invariably close to the golden ratio appear in phyllotaxis (the growth of plants). This phenomenon has to do with the extremely slow convergence of the Fibonacci sequence to the golden ratio. In music, James Tenney's piece *For Ann (rising)* employs the divine ratio explicitly in such a way as having each tone start so it is the golden ratio below the previous tone.

Thus the transcendence of φ is clear. (In the philosophical sense; though it is also true mathematically, given that φ is a transcendental number.) Mark Livio, in his *The Golden Ratio: The Story of Phi, The world's Most Astonishing Number*, sums up the golden ratio in a single, divine sentence: "It is probably fair to say that the Golden Ratio has inspired thinkers of all disciplines like no other number in the history of mathematics."

Bibliography

1. (2006) *The Best of Astraea: 17 Articles on Science, History and Philosophy*. Astrea Web Radio.
2. Dunlap, Richard A. *The Golden Ratio and Fibonacci Numbers*. World Scientific Publishing. 1997.
3. Fibonacci Number – from Wolfram MathWorld.
4. The Golden Ratio. *The MacTutor History of Mathematics archive*.
5. Huntley, H. E. *The Divine Proportion*. 1970.
6. Katz, Victor J. *A History of Mathematics: An Introduction*. Second Edition. March 6, 1998.
7. Knott, Ron. “Fibonacci’s Rabbits.” University of Surrey School of Electronics and Physical Sciences.
8. Livio, Mark. *The Golden Ratio: The Story of Phi. The World’s Most Astonishing Number*. 2002.
9. Weisstein, Eric W. *Golden Ratio Conjugate* at MathWorld.